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An Experimenter's Guide to the Helicity Formalism

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ABSTRACT

The helicity formalism is developed in detail and used to obtain final state angular distributions in relativistic scattering and decay processes. The topics covered include properties of rotation operators, construction of helicity states, scattering and decay amplitudes, the parity of helicity states, and the treatment of identical particles.

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B.1 Introduction

This paper is a pedagogical guide to the helicity formalism, which is the preferred method for obtaining angular distributions in most relativistic scattering and decay processes. The first question that arises is: Why don't we use the spin-orbit formalism that was developed in non-relativistic quantum mechanics? After all, total angular momentum is always conserved, and one should still be able to obtain the total angular momentum operator simply by adding the orbital and spin angular momentum operators for the particles. The problem is that these operators are defined in reference frames that are not at rest with respect to one another. The orbital angular momentum operator is defined in the center of mass (CM) frame, whereas the spin operators are defined in the rest frames of the particles. This leads to some technical problems in describing the spin states which, however, can be overcome (70). The helicity formalism is well suited to relativistic problems because the helicity operator $h = \vec{S} \cdot \hat{p}$ is invariant under both rotations and boosts along $\hat{p}$. As a consequence, one can construct relativistic basis vectors that are either eigenstates of total angular momentum and helicity, or of linear momentum and helicity.

In preparing this paper the author relied on the original paper of Jacob and Wick (71), as well as treatments by Chung (72), Perl (73), Lifshitz and Pitaevskii (74), Martin and Spearman (75), and Jackson (76). The phase conventions are the same as those used by Jacob and Wick.

Before plunging into the details, it is useful to give a brief overview of the main ideas. Consider a decay process, $\alpha \rightarrow 1 + 2$, where α has spin J and spin-projection M along an arbitrarily defined z-axis. We choose the rest frame of α, in which its state vector is $|J,M\rangle$. The amplitude for the final state particles 1,2 to have momenta $\vec{p}_1 = \vec{p}_f$ and $\vec{p}_2 = -\vec{p}_f$ and helicities λ_1,λ_2 is

$$A = \langle \vec{p}_1 = \vec{p}_f, \lambda_1 \,;\; \vec{p}_2 = -\vec{p}_f, \lambda_2 | U | JM \rangle \tag{B.1.1}$$

The final state is referred to as a two-particle plane-wave helicity state. Here U is the time-evolution operator that propagates the initial state through the interaction. Because particles 1,2 have equal and opposite momenta in the CM frame, we can characterize the final state by the direction $\hat{n}(\theta,\varphi)$ of the decay axis with respect to the z-axis (spin-quantization axis of α), by the magnitude p of either particle's momentum, and by the helicities λ_1,λ_2. Thus (suppressing p because it is fixed)

$$A = \langle \theta,\varphi,\lambda_1,\lambda_2 \, | U | JM \rangle \tag{B.1.2}$$

Because $|A|^2$ is the probability for the particles to emerge with polar angles θ,φ, if we can calculate (B.1.2), we have the angular distribution. Typically the experiment does not measure the helicities λ_1,λ_2, so they must be summed over.

The key idea in the helicity formalism is that rotational invariance of the helicities allows one to define a set of two-particle basis states $|j,m,\lambda_1,\lambda_2\rangle$ that have definite total angular momentum j, angular momentum projection m, and helicities λ_1,λ_2. We can then exploit conservation of angular momentum by inserting a complete set of these states into Eq. (B.1.2).

$$\begin{aligned} A &= \sum_{j,m} \langle \theta,\varphi,\lambda_1,\lambda_2 | j,m,\lambda_1,\lambda_2 \rangle \langle j,m,\lambda_1,\lambda_2 | U | JM \rangle \\ &= \sum_{jm} \langle \theta,\varphi,\lambda_1,\lambda_2 | j,m,\lambda_1,\lambda_2 \rangle \delta_{mM} \delta_{jJ} A_{\lambda_1\lambda_2} \\ &= \langle \theta,\varphi,\lambda_1,\lambda_2 | J,M,\lambda_1,\lambda_2 \rangle A_{\lambda_1\lambda_2} \end{aligned} \tag{B.1.3}$$

It will be shown that

$$A = \text{constant} \times D^{J*}_{M\lambda}\,(\varphi,\theta,-\varphi) A_{\lambda_1\lambda_2} \tag{B.1.4}$$

where $\lambda = \lambda_1 - \lambda_2$. This result has a simple interpretation. For a decaying particle α with spin projection M along z, the decay amplitude is equal to the amplitude for its spin to have projection $\lambda = \lambda_1 - \lambda_2$ along the decay axis $\hat{n}(\theta,\varphi)$, multiplied a constant $A_{\lambda_1\lambda_2}$ giving the coupling to the final state helicities.

The plan of this paper is to derive some properties of rotation operators, to use them to construct the plane-wave and total angular momentum helicity states mentioned above, and then to calculate the scattering and decay angular distributions. Finally, parity and the treatment of identical particles in the helicity formalism are discussed.

B.2 Rotation Operators and the $D^J_{M'M}(\alpha\beta\gamma)$ Functions

In this section we derive several results that will be required for the development of the helicity formalism. Especially important are the $D^J_{M'M}(\alpha\beta\gamma)$ functions, which are matrix elements of the rotation operator $R(\alpha\beta\gamma)$ between angular momentum eigenstates.

B.2.1 The Rotation Operator $R(\alpha\beta\gamma)$

We adopt the active view of rotations in which the Cartesian coordinate axes xyz are fixed, and the physical system is rotated with respect to them. The rotation is specified by attaching another coordinate system XYZ to the physical system and measuring the Euler angles of XYZ with respect to the xyz axes. Referring to Fig. B.1, we see that an arbitrary rotation $R(\alpha\beta\gamma)$ can be constructed from 3 successive rotations: 1) a rotation about the z-axis by an angle α, taking Oy into Ou; 2) a rotation about the u-axis by an angle β, taking Oz into OZ; and finally, 3) a rotation about the Z-axis by γ, taking Ou into OY. The complete rotation is therefore

$$R(\alpha\beta\gamma) = R_Z(\gamma)R_u(\beta)R_z(\alpha) = e^{-i\gamma J_Z}e^{-i\beta J_u}e^{-i\alpha J_z} \tag{B.2.1}$$

where we have used the fact that a rotation about a given axis $\hat{n}$ is generated by the angular momentum operator $\vec{J}\cdot\hat{n}$. Equation (B.2.1) for the rotation operator is not very useful because it is not expressed in terms of rotations about the original coordinate axes xyz. To do this, recall that if $|a\rangle$ is a state vector representing some physical system and Q is an observable, then under a rotation R of both the system and observable

$$|a\rangle \rightarrow |a'\rangle = R|a\rangle \tag{B.2.2}$$

and

$$\langle a|Q|a\rangle = \langle a'|Q'|a'\rangle = \langle a|R^\dagger Q' R|a\rangle$$

so that

$$Q' = RQR^\dagger$$

Applying this rule to the sequence of three rotations that make up $R(\alpha\beta\gamma)$ we have

$$J_u = R_z(\alpha)J_y R_z^\dagger(\alpha) \tag{B.2.3}$$

$$J_Z = [R_u(\beta)R_z(\alpha)]J_z[R_u(\beta)R_z(\alpha)]^\dagger = R_u(\beta)J_z R_u^\dagger(\beta)$$

Substituting these expressions into Eq. (B.2.1) gives

$$R(\alpha\beta\gamma) = \left[R_u(\beta)e^{-i\gamma J_z}R_u^\dagger(\beta)\right][R_u(\beta)][R_z(\alpha)]$$

$$= \left[(R_z(\alpha)e^{-i\beta J_y}R_z^\dagger(\alpha))e^{-i\gamma J_z}R_u^\dagger(\beta)\right][R_u(\beta)][R_z(\alpha)]$$

$$R(\alpha\beta\gamma) = e^{-i\alpha J_z}e^{-i\beta J_y}e^{-i\gamma J_z} \tag{B.2.4}$$

In deriving Eq. (B.2.4) we have exploited the unitarity of the rotation operators, so that, for example

$$\exp[-i\gamma J_Z] = \exp[-i\gamma(R_u(\beta)J_z R_u^\dagger(\beta))]$$

$$= R_u(\beta)\exp[-i\gamma J_z]R_u^\dagger(\beta) \tag{B.2.5}$$

The expression (B.2.4) is important because it expresses an arbitrary rotation specified by Euler angles (α,β,γ) in terms of rotations about the fixed axes xyz. It appears in almost every paper on the helicity formalism, and, as we will see, it is the origin of the $D^J_{M'M}(\alpha\beta\gamma)$ functions.

B.2.2 Representations of Rotation Operators

It is useful to evaluate explicit matrix representations of R. We will denote a matrix representation of the unitary operator R by $\underset{\sim}{R}$.

B.2.2.1 Rotation of Vectors

A 3×3 representation can be obtained by considering rotations of vectors. The effect of a rotation is to take the unit vectors $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ that point along Ox, Oy, and Oz, respectively, into three new unit vectors $\hat{E}_1$, $\hat{E}_2$, $\hat{E}_3$ that point along OX, OY, and OZ. Thus

$$\hat{E}_j = R[\hat{e}_j] \qquad j = 1,2,3 \tag{B.2.6}$$

Expressing $\hat{E}_j$ in terms of the $\hat{e}_j$ basis vectors we have

$$\hat{E}_j = \sum_{i=1}^{3} \hat{e}_i \underset{\sim}{R}_{ij} \tag{B.2.7}$$

To evaluate the matrix elements $\underset{\sim}{R}_{ij}$ we use orthogonality of the $\hat{e}_i$

$$\hat{e}_k \cdot \hat{E}_j = \sum_{i=1}^{3} \hat{e}_k \cdot \hat{e}_i \underset{\sim}{R}_{ij} = \underset{\sim}{R}_{kj} \tag{B.2.8}$$

$$\underset{\sim}{R}_{ij} = \hat{e}_i \cdot \hat{E}_j$$

One can calculate $\hat{e}_i \cdot \hat{E}_j$ in terms of the Euler angles ($\alpha\beta\gamma$).

Because we can write any vector $\vec{V} = V_j\hat{e}_j$, $\vec{V}$ transforms as

$$\vec{V} \rightarrow \vec{V}\,' = R[\vec{V}] = \sum_j V_j R[\hat{e}_j] = \sum_j V_j \hat{E}_j = \sum_{i,j} V_j \hat{e}_i \underset{\sim}{R}_{ij} \tag{B.2.9}$$

so that the components of $\vec{V}\,'$ expressed in the xyz coordinate system are

$$V_i' = \sum_j \underset{\sim}{R}_{ij} V_j \qquad \text{(B.2.10)}$$

Note the difference between the rule for transforming a basis vector (Eq. B.2.7) and the rule for obtaining the new components of a vector (Eq. B.2.10): the indices on $\underset{\sim}{R}$ are interchanged with respect to the summing index.

B.2.2.2 Rotation of Angular Momentum Eigenstates

The angular momentum eigenstates $|jm\rangle$ transform irreducibly under rotations because $[R,J^2] = 0$. Thus, a representation is labeled by the total angular momentum j. The action of $R(\alpha\beta\gamma)$ on the basis state $|jm\rangle$ is

$$R(\alpha\beta\gamma)|jm\rangle = \sum_{m'=-j}^{j} D^j_{m'm}(\alpha\beta\gamma)|jm'\rangle \qquad \text{(B.2.11)}$$

$$\langle jm''|R(\alpha\beta\gamma)|jm\rangle = \sum_{m'=-j}^{j} D^j_{m'm}(\alpha\beta\gamma)\langle jm''|jm'\rangle = D^j_{m''m}(\alpha\beta\gamma)$$

These equations are analogous to Eqs. (B.2.7) and (B.2.8). They express the rotated state in terms of the original basis vectors.

Now we use the expression (B.2.4) to calculate $D^j_{mm'}(\alpha\beta\gamma)$

$$D^j_{m'm}(\alpha\beta\gamma) = \langle jm'|e^{-i\alpha J_z}e^{-i\beta J_y}e^{-i\gamma J_z}|jm\rangle$$

Thus

$$D^j_{m'm}(\alpha\beta\gamma) = e^{-i\alpha m'} d^j_{m'm}(\beta) e^{-i\gamma m} \qquad \text{(B.2.12)}$$

where

$$d^j_{m'm}(\beta) = \langle jm'|e^{-i\beta J_y}|jm\rangle$$

This matrix element is given by the Wigner formula

$$d^j_{m'm}(\beta) = \sum_n \left\{ \frac{(-1)^n [(j+m)!(j-m)!(j+m')!(j-m')!]^{1/2}}{(j-m'-n)!(j+m-n)!(n+m'-m)!n!} \right. \tag{B.2.13}$$

$$\left. \times (\cos\tfrac{1}{2}\beta)^{2j+m-m'-2n}(-\sin\tfrac{1}{2}\beta)^{m'-m+2n} \right\}$$

The sum includes all integers n for which all of the arguments of the factorials are positive. Although Eq. (B.2.13) is somewhat complicated, the $d^j_{m'm}(\beta)$ have many simple properties. Clearly these functions are real and from Eq. (B.2.13)

$$d^j_{m'm}(-\beta) = (-1)^{m'-m} d^j_{m'm}(\beta) \tag{B.2.14}$$

Also, $R^\dagger = R^{-1}$ implies $\langle jm|R|jm'\rangle = \langle jm'|R^{-1}|jm\rangle^*$ so that

$$\left[D^j_{mm'}(0,\beta,0)\right] = \langle jm|e^{-i\beta J_y}|jm'\rangle = \langle jm'|e^{i\beta J_y}|jm\rangle \tag{B.2.15}$$

$$= D^j_{m'm}(0,-\beta,0)$$

where we have used the reality of the $d^j_{m'm}(\beta)$ functions. Thus

$$d^j_{m'm}(-\beta) = d^j_{mm'}(\beta) \tag{B.2.16}$$

Using (B.2.14) and (B.2.16), we find that

$$d^j_{m'm}(\beta) = (-1)^{m'-m} d^j_{mm'}(\beta)$$

and from (B.2.16) we have

$$D^j_{m'm}(\alpha\beta\gamma) = e^{-im'\alpha} d^j_{m'm}(\beta) e^{-im\gamma} = e^{-im\gamma} d^j_{mm'}(-\beta) e^{-im'\alpha}$$

$$= D^j_{mm'}(\gamma,-\beta,\alpha) \tag{B.2.17}$$

From Eq. (B.2.13) we can calculate $d^j_{m'm}(\beta)$ for $\beta = \pi, 2\pi$.

$$d^j_{m'm}(\pi) = (-1)^{j-m}\,\delta_{m',-m}$$

$$d^j_{m'm}(2\pi) = (-1)^{2j}\,d^j_{m'm}(0) = (-1)^{2j}\,\delta_{m'm} \qquad \text{(B.2.18)}$$

Finally, there is the extremely useful orthogonality relation

$$\int_0^{2\pi} d\alpha \int_0^{2\pi} d\gamma \int_0^{\pi} \sin\beta\, d\beta \left[D^{j*}_{mn}(\alpha\beta\gamma) D^{j'}_{m'n'}(\alpha\beta\gamma)\right] = \frac{8\pi^2}{2j+1}\,\delta_{mm'}\delta_{nn'}\delta_{jj'} \qquad \text{(B.2.19)}$$

B.3 Plane-Wave Helicity States

B.3.1 One-Particle Plane-Wave Helicity States

The definition of one-particle plane-wave helicity states is intuitive. In the case of massive particles we begin with the rest state $|\vec{p} = O, s, \lambda\rangle$, which has spin s and spin projection λ along the z-axis. In the rest frame the spin projection and the helicity are equivalent. But when this state is rotated only the helicity $\lambda = \vec{s} \cdot \hat{p}$ remains invariant, and we will use it to label the state. Physically, the invariance is due to the fact that the quantization axis $\hat{p}$ rotates along with the spin $\vec{s}$ of the system.

To obtain the state $|\vec{p}, s, \lambda\rangle$, we first rotate $|\vec{p} = 0, s, \lambda\rangle$ so that its quantization axis points along $\hat{p}(\theta,\varphi)$ and then apply a Lorentz boost along $\hat{p}(\theta,\varphi)$.

$$|\vec{p}, s, \lambda\rangle = L(\vec{p}) R(\alpha = \varphi, \beta = \theta, \gamma = -\varphi) |\vec{p} = 0, s, \lambda\rangle \qquad \text{(B.3.1)}$$

The choice $\gamma = -\varphi$ is conventional (Jacob and Wick, Ref. (71)), and has no physical meaning. It is convenient because as $\theta \to 0$

$$R(\varphi, \theta, -\varphi) |\vec{p} = 0, s, \lambda\rangle = \sum_{M'} D^s_{M'\lambda}(\varphi, \theta = 0, -\varphi) |\vec{p} = 0, s, M'\rangle$$

$$= \sum_{M'} e^{-i\varphi M'} \delta_{M'\lambda} e^{i\varphi\lambda} |\vec{p} = 0,s,M'\rangle$$

$$= |\vec{p} = 0,s,\lambda\rangle \tag{B.3.2}$$

independently of φ. The notation $L(\vec{p})$ means: boost the rest-frame particle with velocity V along $\hat{p}(\theta,\varphi)$ such that its final momentum is $\vec{p}$.

This procedure is completely equivalent to one in which we first boost along the z-axis and then rotate to the $\hat{p}(\theta,\varphi)$ direction, because

$$|\vec{p},s,\lambda\rangle = L(\vec{p})R(\varphi,\theta,-\varphi)|\vec{p} = 0,s,\lambda\rangle$$

$$= R(\varphi,\theta,-\varphi)[R^{-1}(\varphi,\theta,-\varphi)L(\vec{p})R(\varphi,\theta,-\varphi)]|\vec{p} = 0,s,\lambda\rangle$$

$$= R(\varphi,\theta,-\varphi)L(\vec{p}_z = p\hat{z})|\vec{p} = 0,s,\lambda\rangle \tag{B.3.3}$$

where we have used

$$L(\vec{p}) = R(\varphi,\theta,-\varphi)L(\vec{p}_z = p\hat{z})R^{-1}(\varphi,\theta,-\varphi)$$

There is one more phase question. To obtain the state $|-\vec{p}_z,s,\lambda\rangle$ we use Eq. (B.3.1) or Eq. (B.3.3) with $\theta = \pi$, but there is no unique choice for φ. The ambiguity can be removed by imposing the desirable condition

$$\lim_{-\vec{p}_z \to 0} |-\vec{p}_z,s,\lambda\rangle = \lim_{\vec{p}_z \to 0} |\vec{p}_z,s,-\lambda\rangle \tag{B.3.4}$$

Because (using Eq. (B.2.18))

$$e^{-i\pi J_y}|\vec{p} = 0,s,\lambda\rangle = \sum_{\lambda'} D^s_{\lambda'\lambda}(0,\pi,0)|\vec{p} = 0,s,\lambda'\rangle$$

$$= \sum_{\lambda'} (-1)^{s-\lambda}\, \delta_{\lambda',-\lambda}|\vec{p} = 0,s,\lambda'\rangle$$

$$= (-1)^{s-\lambda}\, |\vec{p} = 0,s,-\lambda\rangle \tag{B.3.5}$$

we have

$$(-1)^{s-\lambda} e^{-i\pi J_y} |\vec{p} = 0,s,\lambda\rangle = |\vec{p} = 0,s,-\lambda\rangle \tag{B.3.6}$$

Comparing with Eq. (B.3.4) we see that the definition is

$$|-\vec{p}_z,s,\lambda\rangle = (-1)^{s-\lambda} e^{-i\pi J_y}\, |\vec{p}_z,s,\lambda\rangle \tag{B.3.7}$$

Equations (B.3.6) and (B.3.7) will be important when we consider the action of the parity operator on helicity states.

Finally, we choose the Lorentz invariant normalization

$$\langle \vec{p}',s',\lambda'|\vec{p},s,\lambda\rangle = (2\pi)^3\, 2E\, \delta^3(\vec{p}'-\vec{p})\, \delta_{s's}\, \delta_{\lambda'\lambda} \tag{B.3.8}$$

There is no difficulty in treating photons in the helicity formalism. For massive states we can go to the rest frame and thereby obtain all the states $|\vec{p} = 0,s,\lambda\rangle$, $\lambda = -s,-s+1,...s$ by applying the angular momentum lowering operator $J_- = J_x - iJ_y$ to $|\vec{p} = 0,s,s\rangle$, but we cannot do this for photons. Instead, the photon helicity states $|\vec{p},\lambda = +1\rangle$ and $|\vec{p},\lambda = -1\rangle$ are related with the help of the parity operator. See Ref. (71) for details.

B.3.2 Two-Particle Plane-Wave Helicity States in the Center of Mass Frame

We now construct states that represent two particles that are in plane-wave states with momenta $\vec{p}_1$ and $\vec{p}_2$. They are simply the direct product states

$$|\vec{p}_1\lambda_1; \vec{p}_2\lambda_2\rangle \equiv |\vec{p}_1,s_1,\lambda_1\rangle \otimes |\vec{p}_2,s_2,\lambda_2\rangle \tag{B.3.9}$$

The spins s_1 and s_2 of the two particles are fixed and will be suppressed. The Lorentz invariant normalization is obvious from (B.3.8)

$$\langle \vec{p}'_1\lambda'_1;\vec{p}'_2\lambda'_2|\vec{p}_1\lambda_1;\vec{p}_2\lambda_2\rangle = (2\pi)^6 4E_1E_2\delta^3(\vec{p}'_1 - \vec{p}_1)\delta^3(\vec{p}'_2 - \vec{p}_2)\delta_{\lambda'_1\lambda_1}\delta_{\lambda'_2\lambda_2} \tag{B.3.10}$$

We now pick the CM frame, so that $\vec{p}_1 = -\vec{p}_2 = \vec{p}$. Because the particles are back to back, we can now specify the <u>same</u> two-particle state in terms of spherical coordinates p,θ,φ, where $p = |\vec{p}_1| = |\vec{p}_2|$ and (θ,φ) are the angles of $\hat{p}_1$. The state vector is written $|p\theta\varphi\lambda_1\lambda_2\rangle$. It is shown in Section B.8 that the normalization expressed in spherical coordinates is

$$\langle p'\theta'\varphi'\lambda'_1\lambda'_2|p\theta\varphi\lambda_1\lambda_2\rangle = (2\pi)^6\frac{4\sqrt{s}}{p}\,\delta^4(\mathbf{P}'^{\alpha} - \mathbf{P}^{\alpha})\delta(\cos\theta' - \cos\theta)\delta(\varphi' - \varphi) \times \delta_{\lambda'_1\lambda_1}\delta_{\lambda'_2\lambda_2} \tag{B.3.11}$$

where $\mathbf{P}^{\alpha} = \mathbf{P}_1^{\alpha} + \mathbf{P}_2^{\alpha}$ is the total 4-momentum in the CM frame

$$\mathbf{P}^{\alpha} = (E,0,0,0) \qquad E = E_1 + E_2 = \sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2} \tag{B.3.12}$$

In effect, we are making a change of variables from $\vec{p}_1$, $\vec{p}_2$ (6 variables = 3 + 3) to $\mathbf{P}^{\alpha}$, $\cos\theta,\varphi$ (6 variables = 4 + 2).

Because the two-particle CM plane-wave states are eigenstates of total four-momentum $\mathbf{P}^{\alpha}$, it is useful to factor out the eigenstate $|\mathbf{P}^{\alpha}\rangle$. From (B.3.11) we see that a convenient factorization is

$$|p\theta\varphi\lambda_1\lambda_2\rangle = (2\pi)^3\left[\frac{4\sqrt{s}}{p}\right]^{1/2} |\theta\varphi\lambda_1\lambda_2\rangle|\mathbf{P}^{\alpha}\rangle \tag{B.3.13}$$

The factorization has been chosen so that the normalization (B.3.11) is preserved if we define the normalization of $|\mathbf{P}^{\alpha}\rangle$ and $|\theta\varphi\lambda_1\lambda_2\rangle$ to be

$$\langle \mathbf{P}'^{\alpha}|\mathbf{P}^{\alpha}\rangle = \delta^4(\mathbf{P}'^{\alpha} - \mathbf{P}^{\alpha}) \tag{B.3.14}$$

$$\langle \theta'\varphi'\lambda'_1\lambda'_2|\theta\varphi\lambda_1\lambda_2\rangle = \delta(\cos\theta' - \cos\theta)\delta(\varphi' - \varphi)\delta_{\lambda'_1\lambda_1}\delta_{\lambda'_2\lambda_2} \tag{B.3.15}$$

We will see later that Eq. (B.3.14) is the source of the four-momentum conserving δ-function that is always factored out of the (momentum conserving) S-matrix.

B.4 Construction of Two-Particle States with Definite Total Angular Momentum: The Two-Particle Spherical-Wave Helicity Basis

To apply conservation of angular momentum to the transition matrix element, it is necessary to use eigenstates of total angular momentum as the basis for our two-particle CM states. These new basis states will be denoted by $|p,J,M,\lambda_1,\lambda_2\rangle$. Here p is the magnitude of the momentum of either particle, J is the total angular momentum of the two-particle system, M is the eigenvalue of J_z, and λ_1,λ_2 are the helicities of the two particles.Note that p,J,λ_1,λ_2 are all invariant under rotations and thus can be specified simultaneously with M. It may seem strange that p appears in the definition of these states, since neither particle is in an eigenstate of $\vec{P}$. But recall that we are specifying the magnitude, not the direction, of $\vec{p}$. A more intuitive label would be the total center of mass energy

$$E = \sqrt{m_1^2 + p^2} + \sqrt{m_2^2 + p^2} \tag{B.4.1}$$

but the use of p is conventional.

Because the $|p,J,M,\lambda_1,\lambda_2\rangle$ states are eigenstates of total angular momentum, they transform irreducibly under rotations

$$|p,J,M,\lambda_1,\lambda_2\rangle \rightarrow \sum_{M'} D^J_{M'M}(\alpha\beta\gamma)|p,J,M',\lambda_1,\lambda_2\rangle \tag{B.4.2}$$

The two particle plane-wave states $|p,\theta,\varphi,\lambda_1,\lambda_2\rangle$, which we defined as the direct products of two one-particle plane-wave states, do not have definite J,M. Under rotations they transform according to a fully reducible representation $\underset{\sim}{R}$ of the rotation group. (To say that $\underset{\sim}{R}$ is fully reducible means that it can be decomposed into a direct sum over all irreducible representations specified by J,M. In less abstract terms, the matrix $\underset{\sim}{R}$ that rotates $|p,\theta,\varphi,\lambda_1,\lambda_2\rangle$ is a block diagonal matrix. Each block is responsible for transforming the components of $|p,\theta,\varphi,\lambda_1,\lambda_2\rangle$ with a particular value of J. Of course, $\underset{\sim}{R}$ and $|p,\theta,\varphi,\lambda_1,\lambda_2\rangle$ are then both written in terms of the $|p,J,M,\lambda_1,\lambda_2\rangle$ basis.) What is the transformation between these two bases? We write the expansion

$$|p,\theta,\varphi,\lambda_1,\lambda_2\rangle = \sum_{J,M} c_{JM}(p,\theta,\varphi,\lambda_1,\lambda_2)|p,J,M,\lambda_1,\lambda_2\rangle \tag{B.4.3}$$

To determine the coefficients c_{JM} we use a trick: it is easy to evaluate them for $\theta = \varphi = 0$. This direction corresponds to the z-axis, along which M is quantized. We have

$$|p,\theta = 0,\varphi = 0,\lambda_1,\lambda_2\rangle = \sum_{JM} c_{JM}(p,\theta = 0,\varphi = 0,\lambda_1,\lambda_2)|p,J,M,\lambda_1,\lambda_2\rangle \tag{B.4.4}$$

Physically, this vector represents two oppositely moving particles in plane-wave states with momenta $\vec{p}_1 = p\hat{z},\vec{p}_2 = -p\hat{z}$ and with helicities λ_1,λ_2. As we have noted before, this type of state has no orbital angular momentum component along the direction $\hat{p}$, because $\vec{L} = \vec{r}\times\vec{p}$. Hence, $|p,\theta = 0,\varphi = 0,\lambda_1,\lambda_2\rangle$ is an eigenstate of J_z with eigenvalue $\lambda = \lambda_1 - \lambda_2$, and the only terms on the right-hand side of Eq. (B.4.4) are those with $M = \lambda$.

$$|p,\theta = 0,\varphi = 0,\lambda_1,\lambda_2\rangle = \sum_J c_{J\lambda}(p,\theta = 0,\varphi = 0,\lambda_1,\lambda_2)|p,J,\lambda,\lambda_1,\lambda_2\rangle \tag{B.4.5}$$

Now we rotate back to the original state

$$\begin{aligned}|p,\theta,\varphi,\lambda_1,\lambda_2\rangle &= R(\varphi,\theta,-\varphi)|p,\theta = 0,\varphi = 0,\lambda_1,\lambda_2\rangle \\ &= \sum_{J,M'} c_{J\lambda}(p,\theta = 0,\varphi = 0,\lambda_1,\lambda_2)D^J_{M'\lambda}(\varphi,\theta,-\varphi)|p,J,M',\lambda_1,\lambda_2\rangle\end{aligned} \tag{B.4.6}$$

Referring back to Eq. (B.4.3), we can read off the coefficients c_{JM}

$$c_{JM}(p,\theta,\varphi,\lambda_1,\lambda_2) = c_{J\lambda}(p,\theta = 0,\varphi = 0,\lambda_1,\lambda_2)D^J_{M\lambda}(\varphi,\theta,-\varphi) \tag{B.4.7}$$

The coefficients $c_{J\lambda}(p,\theta = 0,\varphi = 0,\lambda_1,\lambda_2)$ are determined up to phase by normalization. It is shown in the appendix that

$$|c_J|^2 = \frac{2J+1}{4\pi} \tag{B.4.8}$$

and we choose

$$c_J = \sqrt{\frac{2J+1}{4\pi}} \tag{B.4.9}$$

The transformation between the two-particle plane-wave helicity basis and the two-particle spherical-wave helicity basis is therefore

$$|p,\theta,\varphi,\lambda_1,\lambda_2\rangle = \sum_{J,M} \sqrt{\frac{2J+1}{4\pi}}\, D^J_{M\lambda}(\varphi,\theta,-\varphi)\,|p,J,M,\lambda_1,\lambda_2\rangle \tag{B.4.10}$$

As an example, take $\theta = \varphi = 0$ and assume spinless particles, so that $\lambda_1 = \lambda_2 = 0$. Then

$$|p,\theta = 0,\varphi = 0,0,0\rangle = \sum_{J,M} \sqrt{\frac{2J+1}{4\pi}}\, D^J_{M0}(0,0,0)\,|p,J,M,0,0\rangle$$

$$= \sum_{J} \sqrt{\frac{2J+1}{4\pi}}\,|p,J,0,0,0\rangle \tag{B.4.11}$$

Finally, we note that because the $|p,J,M,\lambda_1,\lambda_2\rangle$ states have total momentum $\vec{P} = 0$, they are eigenstates of total four-momentum $\mathbf{P}^\alpha$. This is simply because we are working in the CM frame. It is therefore useful to factor out the $|\mathbf{P}^\alpha\rangle$ part of the vector as in Eq. (B.3.13).

$$|p,J,M,\lambda_1,\lambda_2\rangle = (2\pi)^3 \left[\frac{4\sqrt{s}}{p}\right]^{1/2} |J,M,\lambda_1,\lambda_2\rangle|\mathbf{P}^\alpha\rangle \tag{B.4.12}$$

Substituting (B.3.13) and (B.4.12) into the transformation equation (B.4.10), we find, because $|\mathbf{P}^\alpha\rangle$ is invariant under rotations,

$$|\theta,\varphi,\lambda_1,\lambda_2\rangle = \sum_{J,M} \sqrt{\frac{2J+1}{4\pi}}\, D^J_{M\lambda}(\varphi,\theta,-\varphi)\,|J,M,\lambda_1,\lambda_2\rangle \tag{B.4.13}$$

To invert Eq. (B.4.10) or Eq. (B.4.13) we use the orthogonality relation between the D^J functions, Eq. (B.2.19), with $\alpha = -\gamma$. Thus

$$|J,M,\lambda_1,\lambda_2\rangle = \sqrt{\frac{2J+1}{4\pi}} \int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\theta \left[D^{J*}_{M\lambda}(\varphi,\theta,-\varphi)\,|\theta,\varphi,\lambda_1,\lambda_2\rangle\right] \tag{B.4.14}$$

An especially useful form is the inner-product

$$\langle J,M,\lambda'_1,\lambda'_2|\theta,\varphi,\lambda_1,\lambda_2\rangle = \delta_{\lambda'_1\lambda_1}\delta_{\lambda'_2\lambda_2} \sqrt{\frac{2J+1}{4\pi}}\, D^J_{M\lambda}(\varphi,\theta,-\varphi) \tag{B.4.15}$$

where

$$\lambda = \lambda_1 - \lambda_2 \ .$$

B.5 Angular Distributions

B.5.1 Two-Body Scattering

We consider now the process $a + b \rightarrow c + d$ in the center of mass frame. Let the helicities of the particles be $\lambda_a,\lambda_b,\lambda_c,\lambda_d$. The initial state particles a,b have momenta $\vec{p}_a = p_i\hat{z}$ and $\vec{p}_b = -p_i\hat{z}$; the final state particles have momenta $\vec{p}_c = \vec{p}_f$ and $\vec{p}_d = -\vec{p}_f$. Thus, the initial and final two-particle plane-wave helicity states are:

$$|i\rangle = |p_i,\theta_i = 0,\varphi_i = 0,\lambda_a,\lambda_b\rangle = (2\pi)^3 \left[\frac{4\sqrt{s}}{p_i}\right]^{1/2} |\theta_i = 0,\varphi_i = 0,\lambda_a,\lambda_b\rangle|\mathbf{P}^o_i\rangle$$

$$|f\rangle = |p_f,\theta_f,\varphi_f,\lambda_c,\lambda_d\rangle = (2\pi)^3 \left[\frac{4\sqrt{s}}{p_f}\right]^{1/2} |\theta_f,\varphi_f,\lambda_c,\lambda_d\rangle|\mathbf{P}^o_f\rangle \tag{B.5.1}$$

The transition amplitude for scattering from $|i\rangle$ to $|f\rangle$ is

$$\langle f|T|i\rangle = (2\pi)^6 4\sqrt{\frac{s}{p_i p_f}}\langle \mathbf{P}_f^\mu|\langle \theta_f,\varphi_f,\lambda_c,\lambda_d|T|0,0,\lambda_a,\lambda_b\rangle|\mathbf{P}_i^\mu\rangle \tag{B.5.2}$$

Because T conserves energy; but may depend on it, we have

$$\langle f|T|i\rangle = (2\pi)^6 \langle \mathbf{P}_f^\mu|\mathbf{P}_i^\mu\rangle\, 4\sqrt{\frac{s}{p_i p_f}}\langle \theta_f,\varphi_f,\lambda_c,\lambda_d|T(s)|0,0,\lambda_a,\lambda_b\rangle$$

$$= (2\pi)^4\, \delta^4(\mathbf{P}_f^\mu - \mathbf{P}_i^\mu)(2\pi)^2\, 4\sqrt{\frac{s}{p_i p_f}}\langle \theta_f,\varphi_f,\lambda_c,\lambda_d|T(s)|0,0,\lambda_a,\lambda_b\rangle \tag{B.5.3}$$

It is conventional when defining the T-matrix to factor out the four-momentum conserving δ-function

$$S_{fi} = \delta_{fi} + i(2\pi)^4\, \delta^4(\mathbf{P}_f^\mu - \mathbf{P}_i^\mu)T_{fi} \tag{B.5.4}$$

We will write (ignoring the i)

$$\langle f|T|i\rangle = (2\pi)^2\, 4\sqrt{\frac{s}{p_f p_i}}\langle \theta_f,\varphi_f,\lambda_c,\lambda_d|T(s)|0,0,\lambda_a,\lambda_b\rangle \tag{B.5.5}$$

We can now exploit conservation of angular momentum by inserting complete sets of the two-particle spherical helicity states and using Eq. (B.4.15)

$$\langle f|T|i\rangle = (2\pi)^2\, 4\sqrt{\frac{s}{p_i p_f}}\sum_{JM}\sum_{J'M'}\langle \theta_f\varphi_f\lambda_c\lambda_d|JM\lambda_c\lambda_d\rangle$$

$$\times \langle JM\lambda_c\lambda_d|T(s)|J'M'\lambda_a\lambda_b\rangle\langle J'M'\lambda_a\lambda_b|\theta_i = 0,\varphi_i = 0,\lambda_a\lambda_b\rangle$$

$$= (2\pi)^2\, 4\sqrt{\frac{s}{p_i p_f}}\sum\sum_{JMJ'M'}\left(\frac{2J+1}{4\pi}\right)^{1/2}\left(\frac{2J'+1}{4\pi}\right)^{1/2} D^{*J}_{M\lambda_f}(\varphi_f,\theta_f,-\varphi_f)$$

$$\times\, \delta_{JJ'}\delta_{MM'}\langle \lambda_c\lambda_d|T^J(s)|\lambda_a\lambda_b\rangle \times D^{J'}_{M'\lambda_i}(0,0,0)$$

$$\langle f|T|i\rangle = (2\pi)^2\, 4\sqrt{\frac{s}{p_i p_f}}\sum_J\left(\frac{2J+1}{4\pi}\right)D^{J*}_{\lambda_i\lambda_f}(\varphi_f,\theta_f,-\varphi_f)\langle\lambda_c\lambda_d|T^J(s)|\quad\lambda_a\lambda_b\rangle \tag{B.5.6}$$

where $\lambda_i = \lambda_a - \lambda_b$ and $\lambda_f = \lambda_c - \lambda_d$. Let

$$T_{\lambda_c\lambda_d;\lambda_a\lambda_b} = \langle\lambda_c\lambda_d|T^J(s)|\lambda_a\lambda_b\rangle \tag{B.5.7}$$

we have

$$\langle f|T|i\rangle = 4\pi\sqrt{\frac{s}{p_i p_f}}\sum_J(2J+1)e^{i[\lambda_i-\lambda_f]\varphi_f}\, d^J_{\lambda_i\lambda_f}(\theta_f)T^J_{\lambda_c\lambda_d;\lambda_a\lambda_b}(s) \tag{B.5.8}$$

The probability for a transition to final state particles with direction θ_f,φ_f, as a function of θ_f,φ_f--the angular distribution--is given by

$$\frac{d\sigma}{d\Omega_f}(\theta_f,\varphi_f) = \alpha|\langle f|T|i\rangle|^2 \tag{B.5.9}$$

The overall constant α comes from phase space factors and can be ignored if one is calculating only the angular distribution.

B.5.2 Two-Body Decays and Sequential Two-Body Decays

We now obtain the decay angular distribution for $a \rightarrow 1 + 2$, where the decaying particle a has mass m_a, spin J, and spin projection M along an arbitrarily chosen z-axis. The final state particles 1,2 have helicities λ_1,λ_2 and momenta $\vec{p}_1 = \vec{p}_f$, $\vec{p}_1 = -\vec{p}_f$. As usual, we work in the CM frame (rest frame of particle a). From Eq. (B.3.13) the two-particle plane-wave helicity final state is

$$|f\rangle = |p_f\theta_f\varphi_f\lambda_1\lambda_2\rangle = (2\pi)^3\left[\frac{4m_a}{p_f}\right]^{1/2}|\theta_f\varphi_f\lambda_1\lambda_2\rangle|\mathbf{P}_f^0\rangle \tag{B.5.10}$$

Here θ_f,φ_f are the polar angles of $\vec{p}_f$. The amplitude for a to decay into the state $|f\rangle$ is

$$A(a \rightarrow f) = (2\pi)^3 \left[\frac{4m_a}{p_f}\right]^{1/2} \langle\theta_f\varphi_f\lambda_1\lambda_2|U|JM\rangle \tag{B.5.11}$$

where the momentum conserving δ-function has been suppressed. We will also ignore the constants in Eq. (B.5.11), as they have no effect on the angular distribution. To exploit conservation of angular momentum we insert the two-particle spherical helicity basis states $|J_fM_f\lambda_1\lambda_2\rangle$ and use Eq. (B.4.15)

$$A(a \rightarrow f) = \langle\theta_f\varphi_f\lambda_1\lambda_2|U|JM\rangle$$

$$= \sum_{J_f,M_f} \langle\theta_f\varphi_f\lambda_1\lambda_2|J_fM_f\lambda_1\lambda_2\rangle\langle J_fM_f\lambda_1\lambda_2|U|JM\rangle$$

$$= \sum_{J_f,M_f} \left(\frac{2J+1}{4\pi}\right)^{1/2} D^{J_f*}_{M_f,\lambda}(\varphi_f,\theta_f,-\varphi_f)\, \delta_{J_f,J}\delta_{M_f,M}\langle\lambda_1\lambda_2|U|M\rangle \tag{B.5.12}$$

The matrix element $\langle\lambda_1\lambda_2|U|M\rangle$ must be rotationally invariant, so it is more precise to write it as $A_{\lambda_1\lambda_2}$ with no M dependence, The amplitude for $a \rightarrow f$ is therefore

$$A(a \rightarrow f) = \left(\frac{2J+1}{4\pi}\right)^{1/2} D^{J*}_{M\lambda}(\varphi_f,\theta_f,-\varphi_f)A_{\lambda_1\lambda_2} \tag{B.5.13}$$

where $\lambda = \lambda_1 - \lambda_2$. The decay probability is, of course, $|A(a \rightarrow f)|^2$, and if the experiment does not measure the final state helicities λ_1,λ_2 they must be summed over. The angular distribution is

$$\frac{d\sigma}{d\Omega_f}(\theta_f,\varphi_f) = \sum_{\lambda_1,\lambda_2} \left|\left(\frac{2J+1}{4\pi}\right)^{1/2} D^{J*}_{M\lambda}(\varphi_f,\theta_f,-\varphi_f)A_{\lambda_1\lambda_2}\right|^2 \tag{B.5.14}$$

The simplest example is the decay to two spinless particles. The angular distribution is

$$\frac{d\sigma}{d\Omega_f}(\theta_f,\varphi_f) = \frac{2J+1}{4\pi} |D^{J*}_{M0}(\varphi_f,\theta_f,-\varphi_f)A_{00}|^2$$

$$= |Y_J^M(\theta_f,\varphi_f)|^2 \; |A_{00}|^2 \tag{B.5.15}$$

because J must be an integer.

The helicity formalism is easily extended to treat sequential two-body decays. For example, the amplitude for the process

$$a \to 1 + 2$$
$$\quad\;\; \lfloor\!\to 3 + 4$$

where the decaying particle has spin J and the final state particles have helicities $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ is, ignoring overall constants,

$$A(a \to f) = \sum_{\lambda_1} \langle \theta_3\varphi_3\lambda_3\lambda_4 | U(1) | s_1, M_1 = \lambda_1 \rangle\langle \theta_1\varphi_1\lambda_1\lambda_2 | U(a) | JM \rangle$$

$$A(a \to f) = \sum_{\lambda_1} D^{s_1 *}_{\lambda_1,\lambda_3-\lambda_4}(\varphi_3,\theta_3,-\varphi_3)\, D^{J*}_{M,\lambda_1-\lambda_2}(\varphi_1,\theta_1,-\varphi_1) B_{\lambda_3\lambda_4} A_{\lambda_1\lambda_2} \tag{B.5.16}$$

Here we have summed over the allowed helicities of the intermediate particle 1 because they cannot be measured. The angles θ_1,φ_1 are measured in the rest frame of particle a, whereas the angles θ_3,φ_3 are measured in the rest frame of particle 1. (However, φ_3 is the same in both frames.) The z-axis in the frame of particle a is the arbitrarily defined spin-quantization axis for M. The z'-axis in the rest frame of particle 1 is not arbitrary. It is the direction of $\vec{p}_1$ in the particle a rest frame, so that the spin projection along z' is $M_1 = \lambda_1$.

B.6 Parity

The spherical helicity states $|p,J,M,\lambda_1,\lambda_2\rangle$ are not eigenstates of parity. However, we can discover their transformation property by starting with the simpler single-particle plane-wave states from which they are constructed. By exploiting parity conservation in strong and EM interactions we can reduce the number of helicity amplitudes by approximately a factor of 2.

B.6.1 Action of the Parity Operator on Single-Particle Plane-Wave Helicity States

In the rest frame of a single particle state the action of the parity operator Π is simply to multiply the state by its parity eigenvalue η

$$\Pi|\vec{p} = O,s,\lambda\rangle = \eta|\vec{p} = O,s,\lambda\rangle \tag{B.6.1}$$

To find the action of Π on the state $|\vec{p}_z,s,\lambda\rangle$ we use the relations for parity transformed operators

$$L(\vec{p}_z) = \Pi L(-\vec{p}_z)\Pi \tag{B.6.2}$$

$$L(\vec{p}_z) = e^{-i\pi J_y} L(-\vec{p}_z) e^{i\pi J_y}$$

Thus

$$\Pi|\vec{p}_z,s,\lambda\rangle = \Pi L(\vec{p}_z)|\vec{p} = 0,s,\lambda\rangle = L(-\vec{p}_z)\Pi|\vec{p} = 0,s,\lambda\rangle$$

$$= \eta\, L(-\vec{p}_z)|\vec{p} = 0,s,\lambda\rangle$$

$$= \eta\, e^{i\pi J_y} L(\vec{p}_z) e^{-i\pi J_y}\, |\vec{p} = 0,s,\lambda\rangle \tag{B.6.3}$$

With Eq. (B.3.5) to calculate the action of $e^{-i\pi J_y}$ we find

$$\Pi|\vec{p}_z,s,\lambda\rangle = \eta(-1)^{s-\lambda}\, e^{i\pi J_y}|\vec{p}_z,s,-\lambda\rangle \tag{B.6.4}$$

To calculate the effect of parity on a particle with $\vec{p}_z = -p\hat{z}$ we can use Eq. (B.3.7) for $|-\vec{p}_z,s,\lambda\rangle$

in terms of $|\vec{p}_z,s,\lambda\rangle$

$$\Pi|-\vec{p}_z,s,\lambda\rangle = (-1)^{s-\lambda}\, e^{-i\pi J_y}\, \Pi|\vec{p}_z,s,\lambda\rangle \tag{B.6.5}$$

where we have used $[\Pi,R] = 0$ for any rotation operator R. After inserting our previous result Eq. (B.6.4) and using Eq. (B.3.7) with $\lambda \to -\lambda$, we find

$$\Pi|-\vec{p}_z,s,\lambda\rangle = \eta\, |\vec{p}_z,s,-\lambda\rangle = \eta(-1)^{s+\lambda}\, e^{i\pi J_y}|-\vec{p}_z,s,-\lambda\rangle \tag{B.6.6}$$

B.6.2 Action of the Parity Operator on Two-Particle CM Plane-Wave Helicity States

Now consider the action of the parity operator on a CM plane-wave state representing two particles with momenta $\vec{p}_1 = p\hat{z}$ and $\vec{p}_2 = -p\hat{z}$.

$$\begin{aligned}\Pi|p,\theta = 0,\varphi = 0,\lambda_1,\lambda_2\rangle &= \Pi_1|\vec{p}_z,s_1,\lambda_1\rangle\Pi_2|-\vec{p}_z,s_2,\lambda_2\rangle \\ &= \eta_1\eta_2(-1)^{s_1+s_2-\lambda_1+\lambda_2}\, e^{i\pi J_y(1)}|\vec{p}_z,s_1,-\lambda_1\rangle\, e^{i\pi J_y(2)}|-\vec{p}_z,s_2,-\lambda_2\rangle \\ &= \eta_1\eta_2(-1)^{s_1+s_2-\lambda_1+\lambda_2}\, e^{i\pi J_y}|p,\theta = 0,\varphi = 0,-\lambda_1,-\lambda_2\rangle \end{aligned} \tag{B.6.7}$$

B.6.3 Action of the Parity Operator on Two-Particle CM Spherical Helicity States

By substituting the spherical helicity state expansion Eq. (B.4.10) for the two-particle plane-wave states on both sides of (B.6.7) we find

$$\Pi\sum_{JM} c_J D^J_{M\lambda}(0,0,0)|p,J,M,\lambda_1,\lambda_2\rangle =$$

$$\eta_1\eta_2(-1)^{s_1+s_2-\lambda_1+\lambda_2}\, e^{i\pi J_y}\sum_{J,M} c_J D^J_{M,-\lambda}(0,0,0)|p,J,M,-\lambda_1,-\lambda_2\rangle \tag{B.6.8}$$

The sums over M are trivial, because

$$D^J_{M\lambda}(0,0,0) = \delta_{M\lambda}$$

To evaluate the right-hand side we use Eq. (B.2.14) and Eq. (B.2.18)

$$e^{i\pi J_y}|p,J,-\lambda,-\lambda_1,-\lambda_2\rangle = \sum_{M'} D^J_{M',-\lambda}(0,-\pi,0)|p,J,M',-\lambda_1,-\lambda_2\rangle$$

$$= (-1)^{J-\lambda}|p,J,\lambda,-\lambda_1,-\lambda_2\rangle \tag{B.6.9}$$

so Eq. (B.6.8) becomes

$$\sum_J c_J \Pi|p,J,\lambda,\lambda_1,\lambda_2\rangle = \eta_1\eta_2(-1)^{s_1+s_2-\lambda_1+\lambda_2}\sum_J c_J(-1)^{J-\lambda}|p,J,\lambda,-\lambda_1,-\lambda_2\rangle \tag{B.6.10}$$

This equation must hold term by term, because states with different J are orthogonal and the parity operator on the left-hand side does not change J. Thus

$$\Pi|p,J,\lambda,\lambda_1,\lambda_2\rangle = \eta_1\eta_2(-1)^{J-s_1-s_2}|p,J,\lambda,-\lambda_1,-\lambda_2\rangle \tag{B.6.11}$$

By applying the raising and lowering operators $J_\pm = J_x \pm iJ_y$ to both sides of Eq. (B.6.11) we can step M from $-J$ to J. Thus

$$\Pi|p,J,M,\lambda_1,\lambda_2\rangle = \eta_1\eta_2(-1)^{J-s_1-s_2}|p,J,M,-\lambda_1,-\lambda_2\rangle \tag{B.6.12}$$

This result is expected: parity changes the sign of the individual helicities, but M is unchanged.

B.6.4 Applications of Parity Conservation to Helicity Amplitudes

For strong and electromagnetic interactions the T-matrix commutes with Π, so for a scattering process $a + b \to c + d$

$$\langle\lambda_d,\lambda_d|T^J|\lambda_a,\lambda_b\rangle = \langle\lambda_c,\lambda_d|\Pi T^J \Pi|\lambda_a,\lambda_b\rangle$$

$$= \frac{\eta_c\eta_d}{\eta_a\eta_b}(-1)^{s_c+s_d-s_a-s_b}\langle -\lambda_c,-\lambda_d|T^J|-\lambda_a,-\lambda_b\rangle \tag{B.6.13}$$

where we have used Eq. (B.6.12) for both the initial and final states.

For the decay process $a \rightarrow 1 + 2$, we have

$$\langle\lambda_1,\lambda_2|U|a\rangle = \langle\lambda_1,\lambda_2|\Pi U\Pi|a\rangle = \eta_1\eta_2\eta_a(-1)^{s_1+s_2-J} \times \langle-\lambda_1,-\lambda_2|U|a\rangle \qquad \text{(B.6.14)}$$

Here J and η_a are the spin and parity of a. The relations (B.6.13) and (B.6.14) reduce the number of independent helicity amplitudes by about a factor of 2. The reader can show trivially that a pseudoscalar cannot decay into two pseudoscalars by a parity conserving interaction.

As an example, consider radiative ψ decay

$$\psi \rightarrow \gamma + X$$

$$\eta_\psi = -1, \ \eta_1 = -1, \ \eta_2 = \eta_X$$

$$J = 1, \ s_1 = 1, \ s_2 = s_X$$

$$\langle\lambda_\gamma,\lambda_X|U|\psi\rangle = \eta_X(-1)^{s_X}\langle-\lambda_\gamma,-\lambda_X|U|\psi\rangle$$

or

$$A_{\lambda_\gamma\lambda_X} = \eta_X(-1)^{s_X}A_{-\lambda_\gamma-\lambda_X} \qquad \text{(B.6.15)}$$

Figure B.2 shows the case $s_X = 2$.

B.6.5 Example: Angular Distribution for $\psi \rightarrow \gamma\eta'$; $\eta' \rightarrow \gamma\rho^\circ$; $\rho^\circ \rightarrow \pi^+\pi^-$

The sequence of two-body decays

$$\psi \rightarrow \gamma_1\,\eta'$$
$$\rightarrow \gamma_2\,\rho^\circ$$
$$\rightarrow \pi^+\pi^-$$

is shown in Fig. B.3. The object is to calculate the angular distribution of the final state particles, which are γ_1, γ_2, π^+, π^-. Let the helicities of the particles be denoted by

ψ decay	η' decay	$\rho°$ decay
$\lambda_{\eta'} = 0$	$\lambda_\rho = 0, \pm 1$	$\lambda_{\pi^+} = 0$
$\lambda_{\gamma_1} = \pm 1$	$\lambda_{\gamma_2} = \pm 1$	$\lambda_{\pi^-} = 0$

In e^+e^- interactions the ψ is produced with spin-projection $M = \pm 1$ along the beam axis (no $M = 0$!). This is a consequence of the QED interaction $e^+e^- \rightarrow$ virtual γ at energies large compared to the electron mass, where electrons couple to positrons of the opposite helicity only. It is therefore convenient to define the z-axis to be along the beam direction. With unpolarized beams there is no φ dependence and the origin of φ is arbitrary. The amplitude to produce the final state particles with given helicities and angles is

$$A(M,\lambda_{\gamma_1},\lambda_{\gamma_2}) = \sum_{\lambda_\rho} D^{*s(\psi)}_{M,\lambda_{\gamma_1}-\lambda_{\eta'}}(\varphi,\theta,-\varphi)\, A_{\lambda_{\gamma_1}\lambda_{\eta'}} \tag{B.6.16}$$

$$\times\, D^{*s(\eta')}_{-\lambda_{\eta'},\lambda_{\gamma_2}-\lambda_\rho}(\varphi',\theta',-\varphi')\, B_{\lambda_{\gamma_2}\lambda_\rho} \times D^{*s(\rho)}_{-\lambda_\rho,\lambda_{\pi^+}-\lambda_{\pi^-}}(\varphi'',\theta'',-\varphi'')\, C_{\lambda_{\pi^+}\lambda_{\pi^-}}$$

It is conventional to define the z' and z'' axes by the momentum directions of γ_1 and γ_2. The spin projection of the $\rho°$ along the z'' is therefore $-\lambda_\rho$. Because the η' has spin $= 0$, $\lambda_{\gamma_2} - \lambda_\rho = 0$ and the $\rho°$ cannot have helicity 0. Mathematically, this follows from the constant $D^{*s(\eta')}$ term, for which $s(\eta') = 0$ forces both subscripts to be 0. Substituting the known values for the spins, we find

$$A(M,\lambda_{\gamma_1},\lambda_{\gamma_2}) = D^{*1}_{M,\lambda_{\gamma_1}}(\varphi,\theta,-\varphi) A_{\lambda_{\gamma_1}} B_{\lambda_{\gamma_2}\lambda_{\gamma_2}} D^{*1}_{-\lambda_{\gamma_2}0}(\varphi'',\theta'',-\varphi'')\, C_{00} \tag{B.6.17}$$

If the photon helicities are not measured, we must sum over final states after squaring the amplitude. The angular distribution is therefore

$$\frac{d^2\sigma}{d\cos\theta_{\gamma_1}\, d\cos\theta''_{\pi^-}} = \tfrac{1}{2} \sum_M \sum_{\lambda_{\gamma_1}\lambda_{\gamma_2}} |A(M,\lambda_{\gamma_1},\lambda_{\gamma_2})|^2 \tag{B.6.18}$$

where we have also averaged over the initial spin states of the ψ. We can ignore the helicity couplings A,B,C because they form one overall factor. This would not have been the case if γ_2 were replaced by a massive particle so that there was an additional coupling to helicity 0. In our example, however, we can relate all couplings in the sum with parity (Eq. (B.6.14)).

$$A_{\lambda_1} \equiv A_{\lambda_1,0} = \eta_\psi \eta_\gamma \eta_{\eta'} (-1)^{s(\gamma)+s(\eta')-s(\psi)} A_{-\lambda_1,0}$$

$$= (-1) A_{-\lambda_1,0}$$

$$B_{1,1} = \eta_{\eta'} \eta_\gamma \eta_\rho (-1)^{s(\gamma)+s(\rho)-s(\eta')} B_{-1,-1}$$

$$= (-1) B_{-1,-1} \tag{B.6.19}$$

The factors of (-1) are irrelevant in the squared amplitudes. Thus,

$$\sum_{\lambda_{\gamma_1},\lambda_{\gamma_2}} |A(M,\lambda_{\gamma_1},\lambda_{\gamma_2})|^2 = (|d^1_{M,1}(\theta)|^2 + |d^1_{M,-1}(\theta)|^2)$$

$$\times (|d^1_{1,0}(\theta'')|^2 + |d^1_{-1,0}(\theta'')|^2 \tag{B.6.20}$$

Because

$$d^1_{1,0}(\theta'') = d^1_{0,-1}(\theta'') = (-1) d^1_{-1,0}(\theta'') = \frac{-\sin\theta''}{\sqrt{2}} \tag{B.6.21}$$

the second factor in Eq. (B.6.20) is just $\sin^2\theta''$. Also

$$\left.\begin{aligned} d^1_{M,1} &= d^1_{-1,-M} = (-1)^{M-1} \; d^1_{-M,-1} = d^1_{-M,-1} \\ d^1_{M,-1} &= d^1_{1,-M} = (-1)^{M+1} \; d^1_{-M,1} = d^1_{-M,1} \end{aligned}\right\} \tag{B.6.22}$$

where we have used the simple properties of the d functions. From Eq. (B.6.22) it is clear that the results for $M = \pm 1$ will be the same. Thus

$$\frac{d^2\sigma}{d\cos\theta_{\gamma_1} d\cos\theta''_{\pi^-}} = \left[\left(\frac{1 + \cos\theta_{\gamma_1}}{2} \right)^2 + \left(\frac{1 - \cos\theta_{\gamma_1}}{2} \right)^2 \right] \sin^2\theta''_{\pi^-}$$

$$= (1 + \cos^2\theta_{\gamma_1}) \sin^2\theta''_{\pi^-} \qquad \text{(B.6.23)}$$

where we have ignored overall constants. It should be emphasized that the angles in this formula are measured in the rest frames of the decaying particles ψ and ρ°. An interesting feature of this result is that there is no azimuthal angle dependence of any kind. This is a consequence of the fact that there were no unmeasurable intermediate helicities to be summed over, and will not, in general, be the case. For example, in $\psi \to \gamma f(1270)$; $f \to \pi^+\pi^-$, the sums over f helicity in the amplitude produce φ' dependence in the angular distribution.

B.6.6 Example: Angular Distribution for $\psi \to$ Vector + Pseudoscalar

The process $e^+e^- \to \psi \to$ Vector + Pseudoscalar provides an example of a nontrivial azimuthal angular dependence. For convenience, we consider $\psi \to \rho^\circ\pi^\circ$; $\rho^\circ \to \pi^+\pi^-$, although there are many other examples ($\varphi\eta$, $\varphi\eta'$, $\omega\eta$, $\omega\eta'$, $\omega\pi^\circ$, ...).

It will be shown below that the $\lambda = 0$ amplitude of the ρ° vanishes by parity conservation, and interference between the $\lambda = +1$ and $\lambda = -1$ amplitudes produces a correlation between the production plane and the decay plane of the ρ°. The production plane is defined by the e^+e^- beam axis (the z-axis) and the ρ° momentum vector (the z'-axis); this plane lies at the azimuthal angle φ about the z-axis in the lab frame. The decay plane is defined by the momentum vector of the π^+ in the ρ° rest frame and the momentum vector of the ρ° in the lab frame. This plane lies at the angle φ' about the z'-axis in the ρ° rest frame. With the phase convention used in this paper, the production plane lies at the azimuthal angle φ in the ρ° rest frame as well as in the lab frame. This follows from the fact that the $x'y'z'$ coordinate axes are obtained from the xyz axes by a rotation with the Euler angles $\alpha = \varphi$, $\beta = \theta$, $\gamma = -\varphi$. Thus, the angle between the production plane and the decay plane, which is the only physically meaningful azimuthal angle, is given by $\Delta\varphi = \varphi' - \varphi$.

The calculation is straightforward. Parity conservation (Eq. B.6.14) provides a relation between the helicity amplitudes

$$A_{\lambda_\rho,\lambda_\pi} = \eta_\psi \eta_\rho \eta_\pi (-1)^{J_\psi - J_\rho - J_\pi} A_{-\lambda_\rho,-\lambda_\pi} = -A_{-\lambda_\rho,-\lambda_\pi} \tag{B.6.24}$$

which implies that $A_{0,0}$ vanishes. The ρ° therefore has helicities $\lambda_\rho = \pm 1$ only. The corresponding amplitudes interfere because the helicities characterize an intermediate state and cannot be determined by any measurement of the final state pions. The ψ, as discussed in the previous example, is produced as an incoherent mixture of $M_\psi = \pm 1$. The decay angular distribution is therefore

$$\frac{dN}{d\Omega} = \sum_{M_\psi} \Big| \sum_{\lambda_\rho} D^{*1}_{M_\psi,\lambda_\rho}(\varphi,\theta,-\varphi) A_{\lambda_\rho,0} \, D^{*1}_{\lambda_\rho,0}(\varphi',\theta',-\varphi') \Big|^2 \tag{B.6.25}$$

where we have used $\lambda_\pi = 0$. By substituting the D functions, one finds (ignoring overall constants)

$$\frac{dN}{d\Omega} = \left\{(1 + \cos^2\theta) + \sin^2\theta \cos 2(\varphi' - \varphi)\right\} \sin^2\theta' \tag{B.6.26}$$

As in the decay $\psi \rightarrow \gamma\eta'$; $\eta' \rightarrow \gamma\rho^\circ$, the absence of helicity zero for the ρ° results in a $\sin^2\theta'$ distribution of the pions in the ρ° rest frame. If one integrates over the azimuthal angle, the ρ° has a $(1 + \cos^2\theta)$ distribution relative to the beam axis.The azimuthal angular dependence is

$$\frac{dN}{d\cos\Delta\varphi} = \left\{1 + \tfrac{1}{2}\cos 2(\varphi' - \varphi)\right\} \tag{B.6.27}$$

after integrating over the polar angles. Figure B.4 is a histogram of $\Delta\varphi = \varphi' - \varphi$ for $\psi \rightarrow \rho^\circ\pi^\circ$ events measured by the Mark III detector. The $\cos 2(\varphi' - \varphi)$ dependence is clearly present, although there are dips in the distribution at $\psi = 0, \pi$, and 2π due to the detector acceptance. At these angles the ρ° production and decay planes are aligned, and the charged pions are more likely to go into the ends of the detector (closer to the beam pipe) where the detection efficiency is low.

The correlation between $\cos\theta$ and $\Delta\varphi$ is shown in Fig. B.5. The acceptance is also poor when $|\cos\theta| \sim 1$, but the approximately uniform horizontal bands at $\Delta\varphi = 0$, π, and 2π are evident, and contrast with the strong horizontal density variation in the bands at $\Delta\varphi = \frac{\pi}{2}$ and $\frac{3\pi}{2}$.

B.7 Symmetrization of Helicity States for Identical Particles

In the helicity formalism is it easy to construct appropriately symmetrized states for the analysis of processes with identical particles. Let P_{12} denote the particle interchange operator, which swaps particles 1 and 2. The action of P_{12} on an identical two-particle plane-wave helicity state aligned along the z-axis is, because $s_1 = s_2 = s$,

$$P_{12}|p,\theta = 0,\varphi = 0,\lambda_1\lambda_2\rangle = P_{12}[|\vec{p}_z,s,\lambda_1,1\rangle|-\vec{p}_z,s,\lambda_2,2\rangle] \tag{B.7.1}$$

The particle labels 1,2 have been added to the one-particle helicity states to indicate which of the two states each particle is in. The quantum numbers λ_1,λ_2 refer to states, not particles. Interchanging the particles gives

$$\begin{aligned} P_{12}|p,\theta = 0,\varphi = 0,\lambda_1\lambda_2\rangle &= |\vec{p}_z,s,\lambda_1,2\rangle|-\vec{p}_z,s,\lambda_2,1\rangle \\ &= (-1)^{s-\lambda_1}\, e^{i\pi J_y(1)}|-\vec{p}_z,s,\lambda_1,2\rangle \\ &\quad \times (-1)^{s-\lambda_2}\, e^{-i\pi J_y(2)}|\vec{p}_z,s,\lambda_2,1\rangle \end{aligned} \tag{B.7.2}$$

Some care is required with half-integer spins, where

$$(-1)^{s-\lambda_1}(-1)^{s-\lambda_2} = (-1)^{2s+\lambda_1+\lambda_2} = (-1)(-1)^{\lambda_1+\lambda_2} = (-1)^{\lambda_2-\lambda_1} \tag{B.7.3}$$

The last expression is also valid for bosons. We note that

$$e^{-i\pi J_y} = e^{i\pi J_y}\, e^{-i2\pi J_y} = e^{i\pi J_y}\,(-1)^{2s} \tag{B.7.4}$$

which says that half-integer spin particles pick up a factor of (-1) when they are rotated by 2π. Using these results in Eq. (B.7.2) gives

$$P_{12}|p,\theta = 0,\varphi = 0,\lambda_1\lambda_2\rangle = (-1)^{2s+\lambda_2-\lambda_1}\, e^{i\pi J_y}\, |-\vec{p}_z,s,\lambda_1,2\rangle|\vec{p}_z,s,\lambda_2,1\rangle$$

$$= (-1)^{2s+\lambda_2-\lambda_1}\, e^{i\pi J_y}|p,\theta = 0,\varphi = 0,\lambda_2,\lambda_1\rangle \tag{B.7.5}$$

where the ordering of the λ's in the state vector is now reversed from the LHS of Eq. (B.7.2).

We can quickly determine the action of P_{12} on spherical two-particle helicity states by comparing Eq. (B.7.5) with the action of the parity operator in Eq. (B.6.7). The only differences are that in the present case, $s_1 = s_2 = s$, there are no parity factors, and now the order of the helicities is reversed. Thus, we can read off the answer from Eq. (B.6.12)

$$P_{12}|p,J,M,\lambda_1\lambda_2\rangle = (-1)^{J-2s}|p,J,M,\lambda_2,\lambda_1\rangle \tag{B.7.6}$$

According to the spin-statistics theorem, states of identical bosons must be even under particle exchange, and states of identical fermions must be odd under particle exchange. Thus, the correctly summarized two-particle spherical helicity states are

$$\left[1 + (-1)^{2s}P_{12}\right]|p,J,M,\lambda_1,\lambda_2\rangle = |p,J,M,\lambda_1,\lambda_2\rangle + (-1)^J|p,J,M,\lambda_2,\lambda_1\rangle \tag{B.7.7}$$

The result is the same for bosons and fermions.

As a simple application, we will show that a massive spin-1 particle cannot decay into two photons. From Eq. (B.7.7), the correctly symmetrized two photon CM state is

$$|\Phi\rangle = |p,J{=}1,M,\lambda_1,\lambda_2\rangle - |p,J{=}1,M,\lambda_2,\lambda_1\rangle \tag{B.7.8}$$

But $|\lambda_1 - \lambda_2| \leqslant J = 1$ because there can be no orbital angular momentum about the decay axis

$(\vec{L} = \vec{r} \times \vec{p})$. For photons, which have no zero helicity component, this implies that $\lambda_1 = \lambda_2$. The two states on the RHS of Eq. (B.7.8) are therefore identical and

$$|\Phi\rangle = 0. \tag{B.7.9}$$

B.8 Normalization of Two-Particle Plane Wave Helicity States

We now show how to normalize the two-particle plane-wave helicity states when they are labeled by $|p,\theta,\varphi,\lambda_1,\lambda_2\rangle$ (Eq. B.3.11). The normalization in the $\vec{p}_1,\vec{p}_2$ coordinates is (Eq. B.3.10)

$$\langle \vec{p}'_1,\lambda'_1;\vec{p}'_2,\lambda'_2 | \vec{p}_1,\lambda_1;\vec{p}_2,\lambda_2\rangle = (2\pi)^6 4E_1E_2\delta^3(\vec{p}'_1 - \vec{p}_1)\delta^3(\vec{p}'_2 - \vec{p}_2)\delta_{\lambda'_1\lambda_1}\delta_{\lambda'_2\lambda_2} \tag{B.8.1}$$

We are assuming that all other labels describing internal quantum numbers such as charge, isospin, strangeness, etc. are identical, so that we don't need additional δ-functions on the right-hand side of Eq. (B.8.1). By specifying the state with spherical coordinates, we are making a change of variables from $\vec{p}_1,\vec{p}_2$ to $\mathbf{P},p,\theta,\varphi$, where $\mathbf{P}$ is the total momentum in the CM and $\vec{p}$ is the momentum of one of the particles. It is easily seen that

$$d^3p_1d^3p_2 = |\mathbf{J}|d^3\mathbf{P}d^3p = d^3\mathbf{P}d^3p = d^3\mathbf{P}p^2dpd^2\Omega \tag{B.8.2}$$

We can find the relation between dp and $d\mathbf{P}^\circ$ from

$$\sqrt{s} = \mathbf{P}^\circ = E_1 + E_2 = \sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2}$$

$$d\mathbf{P}^\circ = pdp\left(\frac{1}{\sqrt{p^2 + m_1^2}} + \frac{1}{\sqrt{p^2 + m_2^2}}\right) = \frac{(E_1 + E_2)pdp}{E_1E_2} = \frac{\sqrt{s}pdp}{E_1E_2}$$

Thus

$$d^3p_1d^3p_2 = d^3\mathbf{P}\left(d\mathbf{P}^\circ \frac{pE_1E_2}{\sqrt{s}}\right) d^2\Omega$$

$$= \frac{pE_1E_2}{\sqrt{s}}\, d^4\mathbf{P}d^2\Omega \tag{B.8.3}$$

This change of variables means that

$$\int d^3p_1d^3p_2\langle\vec{p}'_1,\lambda'_1;\vec{p}'_2,\lambda'_2|\vec{p}_1,\lambda_1;\vec{p}_2\lambda_2\rangle = \int d^4\mathbf{P}d^2\Omega\frac{pE_1E_2}{\sqrt{s}}\langle p',\theta',\lambda'_1,\lambda'_2|p,\theta,\varphi,\lambda_1,\lambda_2\rangle$$

Using Eq. (B.8.1), we see that

$$(2\pi)^6 4E_1E_2\delta_{\lambda'_1\lambda_1}\delta_{\lambda'_2\lambda_2} = \int d^4\mathbf{P}d^2\Omega\frac{pE_1E_2}{\sqrt{s}}\langle p',\theta',\lambda'_1,\lambda'_2,|p,\theta,\varphi,\lambda_1,\lambda_2\rangle \tag{B.8.4}$$

which implies

$$\langle p',\theta',\varphi',\lambda'_1,\lambda'_2|p,\theta,\varphi,\lambda_1,\lambda_2\rangle = (2\pi)^6\frac{4\sqrt{s}}{p}\delta^4(\mathbf{P}'^{\alpha}-\mathbf{P}^{\alpha})\delta^2(\Omega'-\Omega)\delta_{\lambda'_1\lambda_1}\delta_{\lambda'_2\lambda_2} \tag{B.8.5}$$

This proves Eq. B.3.11. By definition

$$\langle J',M',\lambda'_1,\lambda'_2|J,M,\lambda_1,\lambda_2\rangle = \delta_{J'J}\delta_{M'M}\delta_{\lambda'_1\lambda_1}\delta_{\lambda'_2\lambda_2} \tag{B.8.6}$$

Using Eq. (B.4.14) with c_J unknown and Eq. (B.3.15) we have

$$\langle J',M',\lambda'_1,\lambda'_2|J,M,\lambda_1,\lambda_2\rangle = |c_J|^2 \int\int d^2\Omega d^2\Omega' D^{J'}_{M'\lambda'}(\varphi',\theta',-\varphi')D^{J*}_{M\lambda}(\varphi,\theta,-\varphi)$$

$$\times\, \delta^2(\Omega' - \Omega)\delta_{\lambda'_1\lambda_1}\delta_{\lambda'_2\lambda_2} \tag{B.8.7}$$

Performing the integral over $d^2\Omega'$ and using orthogonality of the D^J functions with $\alpha = -\gamma = \varphi$ (Eq. B.2.19) gives

$$\langle J',M',\lambda'_1,\lambda'_2|J,M,\lambda_1\lambda_2\rangle = \frac{4\pi}{2J+1}\, |c_J|^2\, \delta_{J'J}\delta_{M'M}\delta_{\lambda'_1\lambda_1}\delta_{\lambda'_2\lambda_2} \tag{B.8.8}$$

By comparing Eq. (B.8.8) with Eq. (B.8.6) we see that

$$|c_J|^2 = \frac{2J+1}{4\pi} \tag{B.8.9}$$

as claimed.