

PHYSICS 115B HWK8

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$$\textcircled{1} \quad \hat{U} = e^{i\hat{\theta}} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \hat{\theta}^n$$

$$\hat{U}^+ = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \hat{\theta}^n \quad (\text{if } \hat{\theta}^T = \hat{\theta}, (\hat{\theta}^n)^T = \hat{\theta}^n)$$

$$\hat{U}^- = e^{-i\hat{\theta}}$$

At this point we could just say

$$\boxed{\hat{U} \hat{U}^+ = e^{i\hat{\theta}} e^{-i\hat{\theta}} = e^0 = 1}$$

but it would be nice to show it explicitly

$$\hat{U} \hat{U}^+ = \sum_{n=0}^{\infty} \frac{i^n}{n!} \hat{\theta}^n \sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \hat{\theta}^m$$

$$\hat{U} \hat{U}^+ = \sum_{n,m=0}^{\infty} \frac{(i^{n+m})}{n! m!} \hat{\theta}^n (-\hat{\theta}^m)$$

Write $p = n + m$

$$\hat{U} \hat{U}^+ = \sum_{p=0}^{\infty} \sum_{n=0}^p \frac{(i)^p}{n! (p-n)!} \hat{\theta}^n (-\hat{\theta})^{p-n}$$

$$\text{Binomial} \quad (a+b)^P = \sum_{n=0}^P \frac{P!}{n!(P-n)!} a^n b^{P-n}$$

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$$\text{Therefore } \hat{U}^\dagger = \sum_{p=0}^{\infty} \frac{(i)^p}{p!} (\hat{0} - \hat{0})^p$$

$$(\hat{0} - \hat{0})^p = (\text{Zero})^p$$

$$\hat{U}^\dagger = \sum_{p=0}^{\infty} \frac{(i \cdot \text{Zero})^p}{p!} = i^0 = 1$$

$$\textcircled{2} \quad |p'\rangle = T(a) |p\rangle = e^{-ia\hat{p}/\hbar} |p\rangle$$

$$|p'\rangle = e^{-ia\hat{p}/\hbar} |p\rangle$$

(Note the distinction between \hat{p} which is an operator and p which is a number)

$$\langle x | p' \rangle = \psi_{p'}(x) = e^{-\frac{iax}{\hbar}} \langle x | p \rangle = e^{-\frac{iax}{\hbar}} \psi_p(x)$$

$$\text{But also } \psi_{p'}(x) = T(a) \psi_p(x) = \psi_p(x-a)$$

$$\text{So } \psi_p(x-a) = e^{-iax/\hbar} \psi_p(x)$$

Now set $x=a$

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$$\psi_p(0) = e^{-i\alpha p/\hbar} \psi_p(a)$$

thus is some constant $= A$

$$\psi_p(a) = A e^{+i\alpha p/\hbar}$$

relabeling $a=x$

$$\boxed{\psi_p(x) = A e^{ipx/\hbar}}$$

$$\begin{aligned} ③ \langle \psi_j | x^2 | \psi_j \rangle &= \frac{2}{a} \int_0^a dx x^2 \sin^2 \frac{j\pi x}{a} \\ &= \frac{2}{a} \frac{Q^3}{12} \left(2 - \frac{3}{\pi j^2} \right) = \frac{Q^2}{6} \left[2 - \frac{3}{\pi^2 j^2} \right] \end{aligned}$$

$$\begin{aligned} \langle \psi_j | x | \psi_j \rangle &= \frac{2}{a} \int_0^a dx x \sin^2 \frac{j\pi x}{a} \\ &= \frac{2}{a} \frac{Q^2}{4} = \frac{Q}{2} \end{aligned}$$

$$\begin{aligned} \langle \psi_j | x | \psi_k \rangle &= \frac{2}{a} \int_0^a x \sin \frac{j\pi x}{a} \sin k \frac{\pi x}{a} dx \\ &= \frac{2}{a} \frac{2Q^2 j k}{\pi^2} \frac{[-1] - 1}{(j^2 - k^2)^2} \\ &= 4Qjk \frac{[-1] - 1}{\pi^2 (j^2 - k^2)^2} \end{aligned}$$

If $j+k$ even

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$$\langle \psi_j | x | \psi_k \rangle = 0$$

If $j+k$ odd

$$\langle \psi_j | x | \psi_k \rangle = -\frac{8a j k}{\pi^2 (j^2 - k^2)^2}$$

~~A~~

(a)

$$\langle \Delta x^2 \rangle_d = \frac{a^2}{6} \left[2 - \frac{3}{\pi^2 j^2} \right] + \frac{a^2}{6} \left[2 - \frac{3}{\pi^2 k^2} \right] - 2 \frac{a}{2} \frac{a}{2}$$

$$\langle \Delta x^2 \rangle_d = a^2 \left[\frac{1}{3} + \frac{1}{3} - \frac{1}{2} \right] - \frac{a^2}{2\pi^2} \left[\frac{1}{j^2} + \frac{1}{k^2} \right]$$

$$\boxed{\langle \Delta x^2 \rangle_d = a^2 \left[\frac{1}{6} - \frac{1}{2\pi^2} \frac{k^2 j^2}{k^2 + j^2} \right]}$$

(b) If $j+k$ even

$$\langle \Delta x^2 \rangle_+ = \langle \Delta x^2 \rangle_d$$

(c) If $j+k$ odd

$$\langle \Delta x^2 \rangle_+ = \langle \Delta x^2 \rangle_d - 2 \frac{64 a^2 j^2 k^2}{\pi^4 (j^2 - k^2)^4}$$

$$\langle \Delta x^2 \rangle_+ = a^2 \left[\frac{1}{6} - \frac{1}{2\pi} \frac{k^2 j^2}{k^2 + j^2} - \frac{128 k^2 j^2}{\pi^4 (j^2 - k^2)^4} \right]$$

(d) If $j+k$ even

$$\langle \Delta x^2 \rangle_- = \langle \Delta x^2 \rangle_+$$

(e) if $j+k$ odd, same solution as (c)
but with the sign in front of the last term flipped

$$\langle \Delta x^2 \rangle_- = a^2 \left[\frac{1}{6} - \frac{1}{2\pi} \frac{k^2 j^2}{k^2 + j^2} + \frac{128 k^2 j^2}{\pi^4 (j^2 - k^2)^4} \right]$$

④ (a) $I=1$ states

$$\bar{\Pi}^+ |11\rangle = \bar{u}\bar{d}$$

$$\bar{\Pi}^0 |10\rangle = \frac{\bar{u}\bar{u} - \bar{d}\bar{d}}{\sqrt{2}}$$

$$\bar{\Pi}^- |1-1\rangle = \bar{d}\bar{u}$$

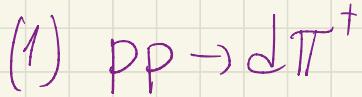
or

$$\begin{matrix} -\bar{u}\bar{d} \\ \bar{d}\bar{d} - \bar{u}\bar{u} \\ -\bar{d}\bar{u} \end{matrix} \over \sqrt{2}$$

$I=0$ state

$$|00\rangle = \frac{\bar{u}\bar{u} + \bar{d}\bar{d}}{\sqrt{2}}$$

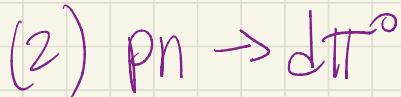
(b)



$$\text{PP} = |1\ 1\rangle$$

$$d\pi^+ = |1\ 1\rangle$$

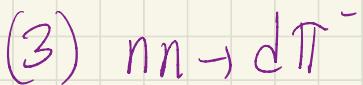
The amplitude $A_{\text{PP}} = A_1 \quad \sigma_{\text{PP}} \sim |A_1|^2$
 where A_I is the amplitude for a given I



$$\text{pn} = |\frac{1}{2}\ \frac{1}{2}\rangle |\frac{1}{2}\ -\frac{1}{2}\rangle = \frac{1}{\sqrt{2}} |1\ 0\rangle + \frac{1}{\sqrt{2}} |0\ 0\rangle$$

$$d\pi^0 = |1\ 0\rangle$$

$$A_{\text{pn}} = \frac{1}{\sqrt{2}} A_1 \quad \sigma_{\text{pn}} \sim \frac{1}{2} |A_1|^2$$



$$\text{nn} = |1\ -1\rangle$$

$$d\pi^- = |1\ -1\rangle \quad A_{\text{nn}} = A_1 \quad \sigma_{\text{nn}} \sim |A_1|^2$$

$$\boxed{\sigma_{\text{PP}} : \sigma_{\text{pn}} : \sigma_{\text{nn}} = 1 : \frac{1}{2} : 1}$$

$$\textcircled{5} \quad T^+(a) \hat{P} T(a) \psi(x)$$

$$= T^+(a) \hat{P} \psi(x-a)$$

$$= T(-a) \hat{P} \psi(x-a)$$

$$= \hat{P} T(-a) \psi(x-a)$$

$$= \hat{P} \psi(x-a+a) = \hat{P} \psi(x)$$

where I used the following

- $T(a) \psi(x) = \psi(x-a)$
- $T^+(a) = T^{-1}(a) = T(-a)$
- $[T(a) \hat{P}] = 0$