

PHYSICS 115B
HOMEWORK 4

page 1

$$(\uparrow) (a) \quad L_x = \frac{1}{2}(L_+ + L_-)$$

$$L_+ |l m\rangle = \hbar \sqrt{l(l+1) - m(m+1)} |l m+1\rangle$$

$$L_- |l m\rangle = \hbar \sqrt{l(l+1) - m(m-1)} |l m-1\rangle$$

Therefore

$$L_x |1 1\rangle = \frac{1}{2} L_- |1 1\rangle = \frac{\hbar}{2} \sqrt{2} |1 0\rangle$$

$$L_x |1 -1\rangle = \frac{1}{2} L_+ |1 -1\rangle = \frac{\hbar}{2} \sqrt{2} |1 0\rangle$$

$$L_x |1 0\rangle = \frac{1}{2} (L_+ + L_-) |1 0\rangle = \frac{\hbar}{2} \sqrt{2} |1 1\rangle + \frac{\hbar}{2} \sqrt{2} |1 -1\rangle$$

$$L_x |1 1\rangle = \frac{\hbar}{\sqrt{2}} |1 0\rangle \quad L_x |1 -1\rangle = \frac{\hbar}{\sqrt{2}} |1 0\rangle$$

$$L_x |1 0\rangle = \frac{\hbar}{\sqrt{2}} (|1 1\rangle + |1 -1\rangle)$$

(b) let the eigenstates be

page 2

$$|u\rangle = a|1\ 1\rangle + b|1\ 0\rangle + c|1\ -1\rangle$$

$$\text{with } a^2 + b^2 + c^2 = 1 \quad (\text{Take } a, b, c \text{ real})$$

$L_x|u\rangle = \lambda|u\rangle$ where λ is the eigenvalue

$$L_x|u\rangle = \frac{\hbar}{\sqrt{2}} [a|1\ 0\rangle + b|1\ 1\rangle + b|1\ -1\rangle + c|1\ 0\rangle]$$

$$L_x|u\rangle = \frac{\hbar}{\sqrt{2}} [b|1\ 1\rangle + (a+c)|1\ 0\rangle + b|1\ -1\rangle]$$

We must have $L_x|u\rangle = \lambda u$, therefore,

$$\frac{\hbar}{\sqrt{2}} b = \lambda a \quad \frac{\hbar}{\sqrt{2}} b = \lambda c \quad \frac{\hbar}{\sqrt{2}} (a+c) = \lambda b \quad (1)$$

One solution is $\lambda=0, b=0, a=-c$ - Call this $|u_0\rangle$

$$|u_0\rangle = a|1\ 1\rangle - a|1\ 0\rangle = \frac{1}{\sqrt{2}} (|1\ 1\rangle - |1\ 0\rangle)$$

where \pm applied the normalization condition $a^2 + b^2 + c^2 = 1$

From equations (1), divide the first one by the second one (which only makes sense if $\lambda \neq 0$)

page 3

$$1 = \frac{a}{c} \text{ gives } a = c$$

which then using the 3rd equation gives $\frac{\lambda b}{\hbar} = \sqrt{2} a$

Put that together with the first of equations (1)

$$\begin{cases} \lambda b / \hbar = \sqrt{2} a \\ \hbar b / \sqrt{2} = \lambda a \end{cases} \quad \begin{cases} \lambda b / \hbar = \sqrt{2} a \\ \hbar b / \lambda = \sqrt{2} a \end{cases}$$

Divide these two equations

$$\left(\frac{\lambda}{\hbar}\right)^2 = 1 \text{ which gives } \lambda = \pm \hbar$$

Next, the normalization condition

$$a^2 + b^2 + c^2 = 1$$

From $a = c$ and $\frac{\lambda b}{\hbar} = \sqrt{2} a$

$$a^2 + (\pm \sqrt{2} a)^2 + a^2 = 1$$

$$4a^2 = 1 \quad a = \frac{1}{2}$$

So the answer is

page 4

Eigenstates of
 L_x

Eigenvalues

$$\frac{1}{2} \left[|1\ 1\rangle + \sqrt{2} |1\ 0\rangle + |1\ -1\rangle \right] = |u_1\rangle \quad +\hbar$$

$$\frac{1}{\sqrt{2}} \left[|1\ 1\rangle - |1\ -1\rangle \right] = |u_0\rangle \quad 0$$

$$\frac{1}{2} \left[|1\ 1\rangle - \sqrt{2} |1\ 0\rangle + |1\ -1\rangle \right] = |u_{-1}\rangle \quad -\hbar$$

$$(c) \quad Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta \quad Y_1^{\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi}$$

$$|u_1\rangle = \frac{1}{2} \sqrt{\frac{3}{8\pi}} \left[-\sin\theta e^{i\phi} + 2\cos\theta + \sin\theta e^{-i\phi} \right]$$

$$|u_1\rangle = \frac{1}{2} \sqrt{\frac{3}{8\pi}} \left[2\cos\theta + \sin\theta (e^{-i\phi} - e^{i\phi}) \right]$$

$$|u_1\rangle = \sqrt{\frac{3}{8\pi}} \left[\cos\theta - i\sin\theta \sin\phi \right] \equiv \sqrt{\frac{3}{8\pi}} \cdot A$$

Next, check that $L_x |u_1\rangle = \hbar |u_1\rangle$

$$\frac{\partial A}{\partial \theta} = \left[-\sin\theta - i\cos\theta \sin\phi \right]$$

$$\text{and } \frac{\partial A}{\partial \phi} = -i\sin\theta \cos\phi$$

$$\text{thus } \underline{L_x |u_1\rangle} = -i\hbar \sqrt{\frac{3}{8\pi}} \left(-\sin\phi \frac{\partial A}{\partial \theta} - \cos\phi \cot\theta \frac{\partial A}{\partial \phi} \right)$$

$$= -i\hbar \sqrt{\frac{3}{8\pi}} \left[+\sin\phi \sin\theta + i\cos\theta \sin^2\phi + i\cos\phi \frac{\cos\theta}{\sin\theta} \sin\theta \cos\phi \right]$$

$$= \hbar \sqrt{\frac{3}{8\pi}} \left[\cos\theta - i\sin\theta \sin\phi \right] = \underline{\underline{\underline{\hbar |u_1\rangle}}} \checkmark$$

Next, work on $|u_0\rangle$

$$|u_0\rangle = \frac{1}{\sqrt{2}} \sqrt{\frac{3}{8\pi}} \left[-\sin\theta e^{i\phi} - \sin\theta e^{-i\phi} \right]$$

$$|u_0\rangle = \sqrt{\frac{3}{16\pi}} \left[-2\sin\theta \cos\phi \right] = -\sqrt{\frac{3}{4\pi}} \sin\theta \cos\phi$$

$$\frac{\partial}{\partial \theta} (\sin\theta \cos\phi) = +\cos\theta \cos\phi$$

$$\text{and } \frac{\partial}{\partial \phi} (\sin\theta \cos\phi) = -\sin\theta \sin\phi$$

$$\text{Therefore } \underline{L_x |u_0\rangle} = +i\hbar \sqrt{\frac{3}{4\pi}} \left[-\sin\phi \cos\theta \cos\phi + \cos\phi \frac{\cos\theta}{\sin\theta} \sin\theta \sin\phi \right] = \underline{\underline{0}} \quad \checkmark$$

Finally, the $|u_{-1}\rangle$ state

$$|u_{-1}\rangle = \frac{1}{2} \sqrt{\frac{3}{8\pi}} \left[-\sin\theta e^{i\phi} - 2\cos\theta + \sin\theta e^{-i\phi} \right]$$

$$|u_{-1}\rangle = -\sqrt{\frac{3}{8\pi}} \left[\cos\theta - i \sin\theta \sin\phi \right] = -\sqrt{\frac{3}{8\pi}} B$$

$$\frac{\partial B}{\partial \theta} = -\sin \theta - i \cos \theta \sin \phi$$

$$\frac{\partial B}{\partial \phi} = -i \sin \theta \cos \phi$$

Then

$$L_x |u_{-1}\rangle = -i\hbar \sqrt{\frac{3}{8\pi}} \left[+\sin \phi \sin \theta + i \cos \theta \sin^2 \phi - \cos \phi \frac{\cos \theta}{\sin \theta} (-i) \sin \theta \cos \phi \right]$$

$$L_x |u_{-1}\rangle = -i\hbar \sqrt{\frac{3}{8\pi}} \left[\sin \theta \sin \phi + i \cos \theta \right] = \underline{\underline{-\hbar |u_{-1}\rangle}}$$

$$2 \quad (a) \quad S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

eigenspinors $\begin{pmatrix} a \\ b \end{pmatrix}$ eigenvalues λ_1 and λ_2

$$\frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\text{Want } \det \begin{pmatrix} -\lambda & -i\hbar/2 \\ i\hbar/2 & -\lambda \end{pmatrix} = 0 \Rightarrow \lambda^2 - \frac{\hbar^2}{4} = 0$$

$$\text{Eigenvalues } \lambda = \pm \frac{\hbar}{2}$$

For the eigenspinors, set $\lambda = \pm \frac{\hbar}{2}$

$$\frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \pm \frac{\hbar}{2} \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow -ib = \pm a$$

Then we have, including the normalization

$$\frac{e^{i\phi}}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \lambda = \frac{\hbar}{2} \quad \text{and} \quad \frac{e^{i\phi}}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \lambda = -\frac{\hbar}{2}$$

(b) Writing the eigenstates from part (a) as y_+ and y_- we would like to write the

state $\chi = c_+ y_+ + c_- y_-$

$$c_+ = y_+^\dagger \chi = \frac{e^{-i\phi}}{\sqrt{2}} (1 - i) \begin{pmatrix} a \\ b \end{pmatrix} = \frac{e^{-i\phi}}{\sqrt{2}} (a - ib)$$

$$c_- = y_-^\dagger \chi = \frac{e^{-i\phi}}{\sqrt{2}} (1 + i) \begin{pmatrix} a \\ b \end{pmatrix} = \frac{e^{-i\phi}}{\sqrt{2}} (a + ib)$$

$$|c_+|^2 = \frac{1}{2} (a - ib)(a^* + ib^*) = \frac{|a|^2 + |b|^2 + i(ab^* - ba^*)}{2}$$

Using $|a|^2 + |b|^2 = 1$

$$|c_+|^2 = \frac{1}{2} + \frac{i}{2} (ab^* - a^*b) = \text{prob of } +\hbar/2$$

$$|c_-|^2 = \frac{1}{2} - \frac{i}{2} (ab^* - a^*b) = \text{prob of } -\hbar/2$$

(1) Always $\frac{\hbar^2}{4}$

page 10

(3) (a) $\chi = A \begin{pmatrix} 3i \\ 4 \end{pmatrix}$

$$\chi^\dagger \chi = 1 = 9|A|^2 + 16|A|^2 = 25|A|^2$$

$$A = \frac{1}{5} e^{i\phi} \quad \text{where } \phi \text{ is an arbitrary phase}$$

(b) Set $\phi = 0$ in the following

$$\langle S_x \rangle = \langle \chi | S_x | \chi \rangle$$

$$\langle S_x \rangle = \frac{1}{25} \frac{\hbar}{2} (-3i \ 4) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix} = \frac{\hbar}{50} (-3i \ 4) \begin{pmatrix} 4 \\ 3i \end{pmatrix}$$

$$\langle S_x \rangle = 0$$

$$\langle S_y \rangle = \frac{\hbar}{50} (-3i \ 4) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix} = \frac{\hbar}{50} (-3i \ 4) \begin{pmatrix} -4i \\ -3 \end{pmatrix}$$

$$\langle S_y \rangle = \frac{\hbar}{50} (-12 - 12) = -\frac{12\hbar}{25}$$

page 11

$$\langle S_z \rangle = \frac{\hbar}{50} (-3i \ 4) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix} = \frac{\hbar}{50} (-3i \ 4) \begin{pmatrix} 3i \\ -4 \end{pmatrix}$$

$$\langle S_z \rangle = \frac{\hbar}{50} (9 - 16) = -\frac{7\hbar}{50}$$

Since $S^2 = \hbar^2 \frac{1}{2}(\frac{1}{2} + 1) \mathbb{I} = \frac{3\hbar^2}{4} \mathbb{I}$ where

\mathbb{I} is the identity matrix,

$$\langle S^2 \rangle = \frac{3\hbar^2}{4}$$

(c) Since $\langle S_x^2 \rangle = \langle S_y^2 \rangle = \langle S_z^2 \rangle = \frac{\hbar^2}{4}$

(always) for spin $\frac{1}{2}$

$$\sigma_{S_x}^2 = \langle S_x^2 \rangle - \langle S_x \rangle^2 = \frac{\hbar^2}{4} - 0 = \frac{\hbar^2}{4}$$

$$\sigma_{S_y}^2 = \frac{\hbar^2}{4} - \left(\frac{12}{25}\right)^2 \hbar^2 = \frac{49}{2500} \hbar^2$$

$$\sigma_{S_z}^2 = \frac{\hbar^2}{4} - \left(\frac{7}{50}\right)^2 \hbar^2 = \frac{576}{2500} \hbar^2 = \frac{144}{625} \hbar^2$$

Taking the square roots next

page 12

$$\sigma_{S_x} = \frac{\hbar}{2} \quad \sigma_{S_y} = \frac{7}{50}\hbar \quad \sigma_{S_z} = \frac{12}{25}$$

④ Three eigenstates

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \chi_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \chi_- = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Use the fact that for operator \hat{O} , its matrix representation is $O_{ij} = \langle i | \hat{O} | j \rangle$

Clearly

$$S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad S^2 = 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(The factor 2 in S^2 comes from $S(S+1) = 2$)

$$\text{From } L_{\pm} |l, m\rangle = \hbar \sqrt{l(l+1) - m(m\pm 1)} |l, m\pm 1\rangle$$

it is easy to build the S_+ and S_- matrices - They are

$$S_+ = \sqrt{2}\hbar \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad S_- = \sqrt{2}\hbar \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{page 13}$$

Then, since

$$S_x = \frac{1}{2}(S_+ + S_-) \quad S_y = \frac{1}{2i}(S_+ - S_-)$$

we can get

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad S_y = \frac{i\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$