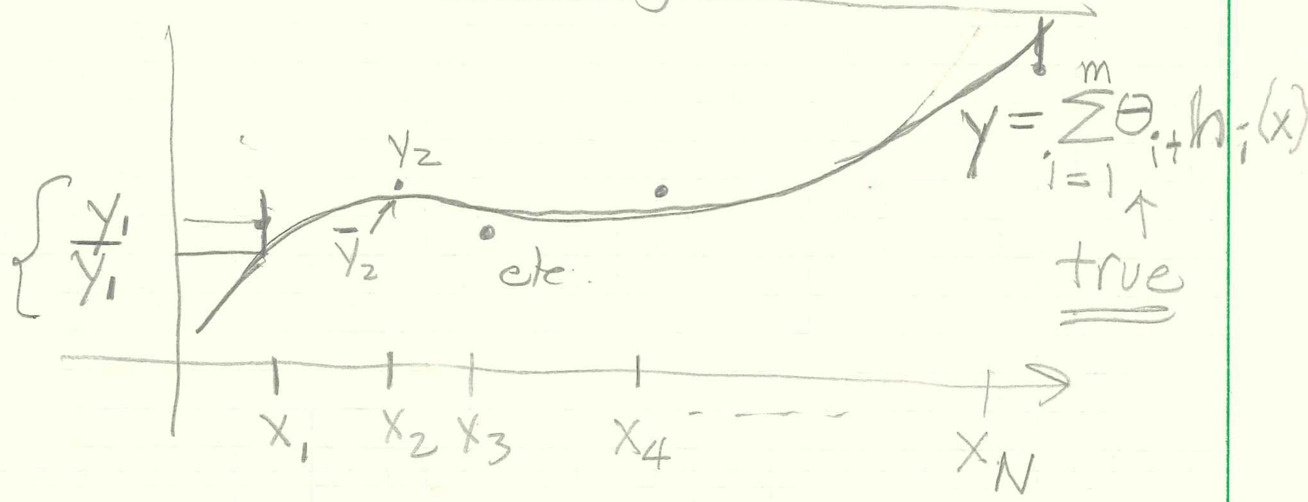
More Formal Fitting + Kalman



$h_i(x)$ = usually polynomial

$h_1(x) = 1$

$h_2(x) = x - x_0$ $\quad x_0$ ... choose in middle

$h_3(x) = x^2 - ax - b$

$\langle y_1^2 \rangle = \bar{Y}_1^2 + \sigma_1^2$ $\quad$ best $x_0 = \dfrac{\sum_{i=1}^{N} \frac{x_i}{\sigma_i^2}}{\sum_{i=1}^{N} \frac{1}{\sigma_i^2}}$

$\langle y_i y_j \rangle = \underbrace{\bar{Y}_i \bar{Y}_j + \sigma_i^2 \delta_{ij}}_{\text{no correlation}}$ $\quad$ more generally, $\bar{Y}_i \bar{Y}_j + V_{ij}$

→ multiple scattering does not satisfy!!

not necessarily true!

$$F(x; \vec{\theta}) = \sum_{j=1}^{m} \theta_j h_j(x)$$

$$\chi^2(y_i; \vec{\theta}) = \sum_{i=1}^{N} \frac{(y_i - F(x_i; \vec{\theta}))^2}{\sigma_i^2} = \sum_{i=1}^{N} \frac{\left(y_i - \sum_{j=1}^{m} \theta_j h_j(x_i)\right)^2}{\sigma_i^2}$$

Big Matrix Fun

$$\left.\frac{\partial \chi^2}{\partial \theta_k}\right|_{\hat{\theta}} = 0 = 2\sum_{i=1}^{N} \frac{\left(y_i - \sum_{j=1}^{m}\hat{\theta}_j h_j(x_i)\right)}{\sigma_i^2}\left(-h_k(x_i)\right)$$

$$\underbrace{\sum_{j=1}^{m}\left\{\sum_{i=1}^{N}\frac{h_k(x_i)h_j(x_i)}{\sigma_i^2}\right\}}_{\text{"Weight Matrix" } m\times m}\hat{\theta}_j = \sum_{i=1}^{N}\frac{h_k(x_i)y_i}{\sigma_i^2}$$

$$\underset{\sim}{W}_{kj} \quad \hat{\theta}_j = Y_k$$

$$\underset{\sim}{W}\,\vec{\hat{\theta}} = \vec{Y}$$

$$\vec{\hat{\theta}} = \underset{\sim}{W}^{-1}\vec{Y}$$

$m \times m$

called ~~c~~covariance matrix $\underset{\sim}{U}$

$$\underset{\sim}{U}^{-1} = \underset{\sim}{W}$$

Good with orthogonal polynomials

$$\sum_{i=1}^{N}\frac{h_k(x_i)h_j(x_i)}{\sigma_i^2} \propto \delta_{kj}$$

(last lecture).

$$\delta\theta_j = \theta_j - \hat{\theta}_j$$

$$\chi^2 = \sum_{i=1}^{N} \frac{\left(y_i - \sum_{j=1}^{m} (\hat{\theta}_j + \delta\theta_j) h_j(x_i)\right)^2}{\sigma_i^2}$$

$$= \sum_{i=1}^{N} \left[\frac{y_i^2}{\sigma_i^2} - 2\frac{y_i}{\sigma_i^2} \sum_{j=1}^{m} (\hat{\theta}_j + \delta\theta_j) h_j(x_i) + \frac{1}{\sigma_i^2} \left(\sum_{j=1}^{m} (\hat{\theta}_j + \delta\theta_j) h_j(x_i) \right)^2 \right]$$

SWAP SUMS!

$$= \sum_{i=1}^{N} \frac{y_i^2}{\sigma_i^2} - 2 \sum_{j=1}^{m} (\hat{\theta}_j + \delta\theta_j) \underbrace{\left[\sum_{i=1}^{N} \frac{y_i h_j(x_i)}{\sigma_i^2} \right]}_{Y_j} + \sum_{j,k=1}^{m} (\hat{\theta}_j + \delta\theta_j) \underbrace{\left[\sum_{i=1}^{N} \frac{h_j(x_i) h_k(x_i)}{\sigma_i^2} \right]}_{\underset{\sim}{W}_{jk}} (\hat{\theta}_k + \delta\theta_k)$$

$$= \sum_{i=1}^{N} \frac{y_i^2}{\sigma_i^2} - 2(\vec{\hat{\theta}} + \delta\vec{\theta}) \cdot \vec{Y} + (\vec{\hat{\theta}} + \delta\vec{\theta})^T \underset{\sim}{W} (\vec{\hat{\theta}} + \delta\vec{\theta})$$

$$+ \underbrace{\vec{\hat{\theta}}^T \underset{\sim}{W} \vec{\hat{\theta}}}_{\vec{\hat{\theta}} \cdot \vec{Y}} + \underbrace{\delta\vec{\theta}^T \underbrace{\underset{\sim}{W} \vec{\hat{\theta}}}_{\vec{Y}} + \vec{\hat{\theta}}^T \underbrace{\underset{\sim}{W} \delta\vec{\theta}}_{\vec{Y}^T}}_{+2\,\delta\vec{\theta} \cdot \vec{Y}} + \delta\vec{\theta}^T \underset{\sim}{W} \delta\vec{\theta}$$

reduces by 1

cancels

$$\underbrace{\chi^2}_{\text{averages to } N} = \underbrace{\sum_{i=1}^{N} \frac{y_i^2}{\sigma_i^2} - \vec{\hat{\theta}} \cdot \vec{Y}}_{\text{so } N-m \text{ best fit}} + \underbrace{\delta\vec{\theta}^T \underset{\sim}{W} \delta\vec{\theta}}_{\text{averages to } m}$$

averages to N

so N − m best fit

averages to m

tells you how much χ^2 changes if you vary your parameters

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$$\hat{\theta}_j = \sum_k (\underset{\sim}{W}^{-1})_{jk} \sum_{i=1}^{N} \frac{h_k(x_i)\, y_i}{\sigma_i^2}$$

$$\delta\hat{\theta}_j = \sum_k (\underset{\sim}{W}^{-1})_{jk} \sum_{i=1}^{N} \frac{h_k(x_i)\, \delta y_i}{\sigma_i^2}$$

$$\delta\vec{\hat{\theta}}^T \underset{\sim}{W}\, \delta\vec{\theta}$$

$$= \sum_{k'm\ell} \sum_{i'=1}^{N} \frac{h_{k'}(x_{i'})\,\delta y_{i'}}{\sigma_{i'}^2} \underbrace{(\underset{\sim}{W}^{-1})_{k'\ell}\, \underset{\sim}{W}_{\ell m}}_{\delta_{k'm}} \sum_{k=1}^{m} (\underset{\sim}{W}^{-1})_{mk} \sum_{i=1}^{N} \frac{h_k(x_i)\,\delta y_i}{\sigma_i^2}$$

$$= \sum_{k,m=1}^{m} \sum_{i'=1}^{N} \frac{h_m(x_{i'})\,\delta y_{i'}}{\sigma_{i'}^2}\, \underset{\sim}{W}^{-1}_{mk} \sum_{i=1}^{N} \frac{h_k(x_i)\,\delta y_i}{\sigma_i^2}$$

$$\langle \delta y_{i'}\, \delta y_i \rangle = \sigma_i^2\, \delta_{ii'}$$

$$\langle \delta\vec{\hat{\theta}}^T \underset{\sim}{W}\, \delta\vec{\theta} \rangle = \sum_{k,m=1}^{m} \underbrace{\sum_{i=1}^{N} \frac{h_m(x_i)\, h_k(x_i)}{\sigma_i^2}}_{W_{mk}} W^{-1}_{mk}$$

$$= \sum_{k,m=1}^{m} W_{mk}\, W^{-1}_{km} = \mathrm{Trace}\overbrace{\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}}^{m}$$

$$= m$$

$$\left\langle \sum_{i=1}^{N} \frac{y_i^2}{\sigma_i^2} - \vec{\hat{\theta}} \cdot \vec{Y} \right\rangle = \sum_{i=1}^{N} \frac{\bar{Y}_i^2}{\sigma_i^2} + N - \vec{\theta}_+ \cdot \vec{Y}_+ - \underbrace{\delta\vec{\hat{\theta}} \cdot \delta\vec{Y}}_{\underbrace{\delta\vec{\hat{\theta}}^T \underset{\sim}{W}\, \delta\vec{\hat{\theta}}}_{-m}}$$

most cancel

$$= N - m$$

$$\langle \delta\theta_i \, \delta\theta_j \rangle$$

$$= \left\langle \left(\sum_k W^{-1}_{ik} \delta y_k\right)^T \left(\sum_\ell W_{j\ell} \delta y_\ell\right) \right\rangle$$

$$\left\langle \sum_{k\ell} \delta y_k W^{-1}_{ki} W^{-1}_{j\ell} \delta y_\ell \right\rangle$$

$$\left\langle \sum_{k\ell m n} \frac{h_k(x_m)\,\delta y_m}{\sigma_m^2} W^{-1}_{ki} W^{-1}_{j\ell} \frac{h_\ell(x_n)\,\delta y_n}{\sigma_n^2} \right\rangle$$

$$\langle \delta y_m \delta y_n \rangle = \sigma_m^2 \delta_{mn}$$

$$\underbrace{\underbrace{\sum_{k\ell m} \underbrace{\frac{h_k(x_m)\, h_\ell(x_m)}{\sigma_m^2}}_{} W^{-1}_{ki}}_{W_{k\ell}} \cdots}_{\delta_{i\ell}} W^{-1}_{j\ell}$$

$$\langle \delta\theta_i \delta\theta_j \rangle = W^{-1}_{ji} = W^{-1}_{ij}$$

Error Envelope

$$y = \sum_{j=1}^{m} \hat{\theta}_j h_j(x)$$

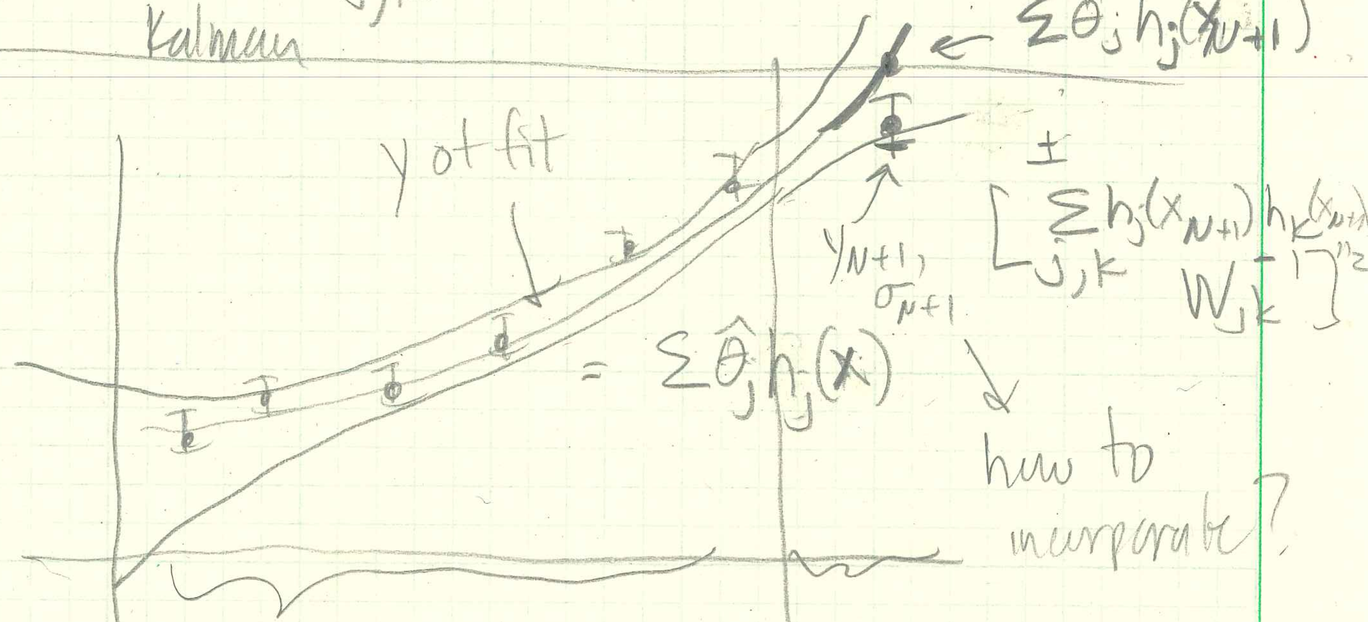
$$\delta y = \sum_{j=1}^{m} \delta\hat{\theta}_j h_j(x)$$

$$\delta y^2 = \left(\sum_{j=1}^{m} \delta\hat{\theta}_j h_j(x)\right)\left(\sum_{k=1}^{m} \delta\hat{\theta}_k h_k(x)\right)$$

$$\langle \delta y^2 \rangle = \sum_{j,k} h_j(x) h_k(x) \underbrace{\langle \delta\theta_j \delta\theta_k \rangle}_{W^{-1}_{jk}}$$

$$\langle \delta y^2 \rangle^{1/2} = \left[\sum_{j,k} h_j(x) h_k(x) W^{-1}_{jk}\right]^{1/2}$$

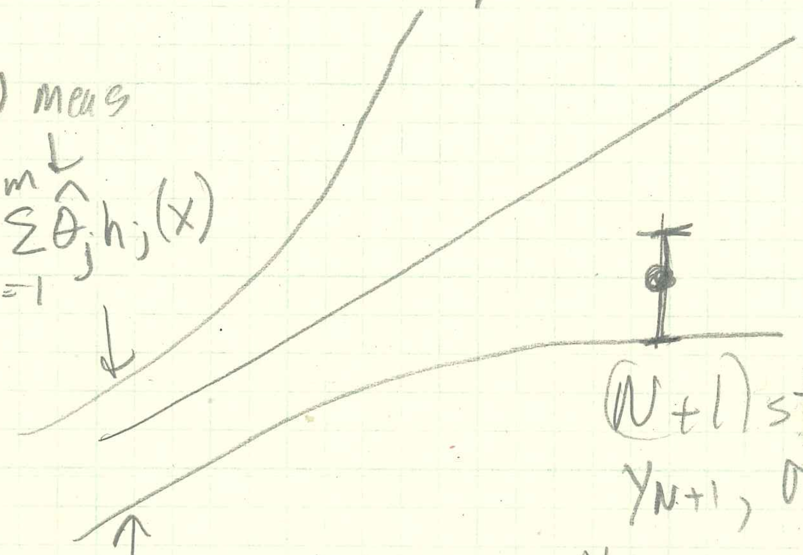
Kalman



"Kalman" concept

N meas

$\sum_{j=1}^{m} \hat{\theta}_j h_j(x)$



$(N+1)$st

y_{N+1}, σ_{N+1}

consistent

$\pm \left[\sum_{jk} h_j(x) h_k(x) W_{jk}^{-1} \right]^{1/2}$

you'd test

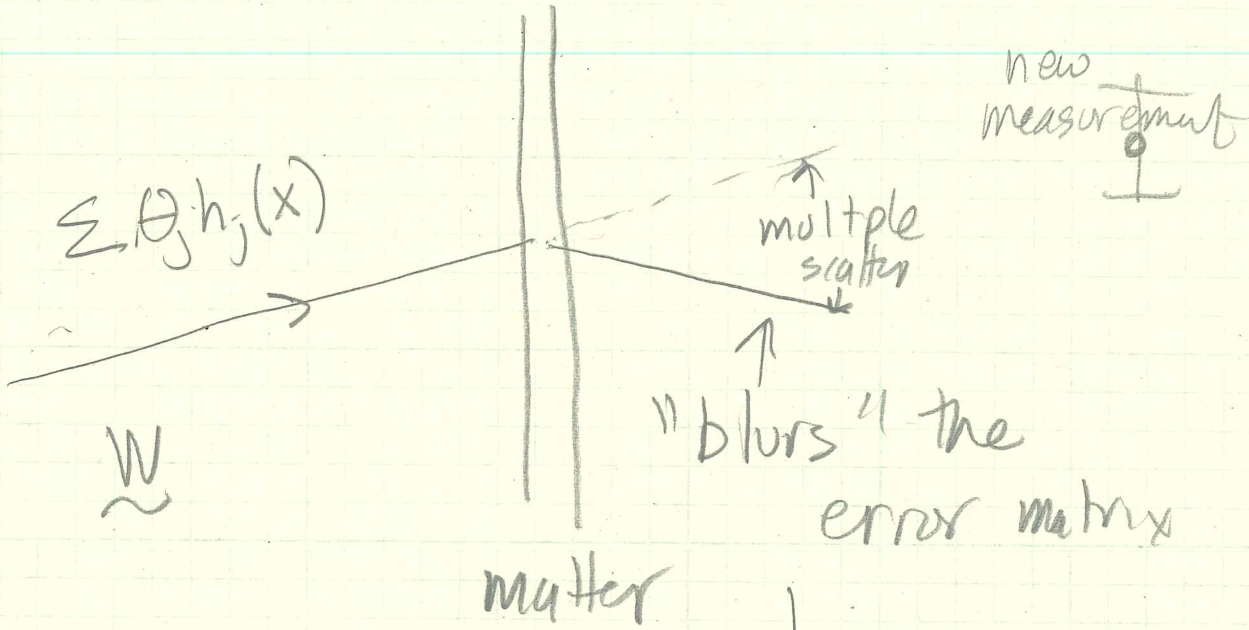
$$\frac{\left(y_{N+1} - \sum_{j=1}^{m} \hat{\theta}_j h_j(x_{N+1}) \right)^2}{\sigma_{N+1}^2 + \sum_{jk} h_j(x)_{N+1} h_k(x)_{N+1} W_{jk}^{-1}}$$

This one point can be used to update the estimates for the $\hat{\theta}_j$

$$\Delta \chi^2 = \delta\hat{\theta}^T \underset{\sim}{W}\, \delta\hat{\theta} + \frac{\left(y_{N+1} - \sum_{j=1}^{m} (\hat{\theta}_j + \delta\hat{\theta}_j) h_j \right)^2}{\sigma_{N+1}^2}$$

from N meas

$$= \sum_{jk} \delta\theta_j W_{jk} \delta\theta_k + \frac{1}{\sigma_{N+1}^2} \Big(y_{N+1}^2 - 2 y_{N+1} \sum_{j=1}^{m} (\hat{\theta}_j + \delta\theta_j) h_j$$
$$+ \sum_{jk} (\hat{\theta}_j + \delta\hat{\theta}_j)(\hat{\theta}_k + \delta\hat{\theta}_k) h_j h_k$$



$$\text{angle} \rightarrow \delta\theta_2 \simeq \frac{14\,\text{MeV}}{\beta c p}\sqrt{\frac{x}{X_0}}$$

$$\underset{\sim}{W} \rightarrow \underset{\sim}{W}^{-1} \Rightarrow \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & \multicolumn{3}{c}{\multirow{3}{*}{$\begin{pmatrix} w^{-1}_{11} & w^{-1}_{12} & w^{-1}_{13} \\ w^{-1}_{21} & w^{-1}_{22} + (\delta\theta_2)^2 & \\ & & \end{pmatrix}$}} \\ 2 & & & \\ 3 & & & \end{array}$$

weight error (covariance)

tracking: $\simeq$ 3 params

blur the weight

reduced blurred weight

$\underset{\sim}{W}'$

update this!

$$\frac{\partial \Delta x^2}{\partial \theta_\ell} = \sum_k W_{\ell k}\,\delta\theta_k + \sum_j W_{j\ell}\,\delta\theta_\ell$$

$$- \frac{2\, y_{N+1}\, h_\ell(x_{N+1})}{\sigma_{N+1}^2}$$

$$+ 2 \sum_k \frac{h_j(x_{N+1})\, h_k(x_{N+1})}{\sigma_{N+1}^2}\, \delta\theta_k = 0$$

$$\Rightarrow \underbrace{\left(\underset{\substack{\uparrow \\ N}}{\underset{\sim}{W}} + \underset{\substack{\uparrow \\ \frac{h_j(x_{N+1})\, h_k(x_{N+1})}{\sigma_{N+1}^2}}}{\delta \underset{\sim}{W}} \right)}_{\text{new weight matrix (more weight)}} \delta\vec{\theta} = \underset{\substack{\uparrow \\ \frac{y_{N+1}\, h_j(x_{N+1})}{\sigma_{N+1}^2}}}{\delta \vec{Y}}$$

$$\delta\vec{\theta} = \underbrace{\left(\underset{\sim}{W} + \delta \underset{\sim}{W} \right)^{-1}}_{\substack{\text{new covariance} \\ \text{less covariance}}} \delta\vec{Y}$$

This is not the full power of the Kalman technique, though.

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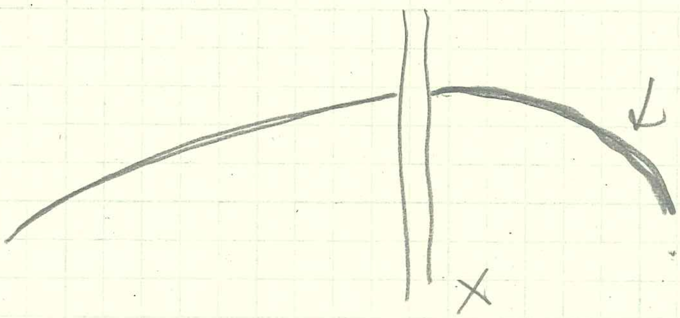
Can be generalized.



θ_3 increased (more curvature).

↑ energy loss

$\delta\theta_3$ blurring

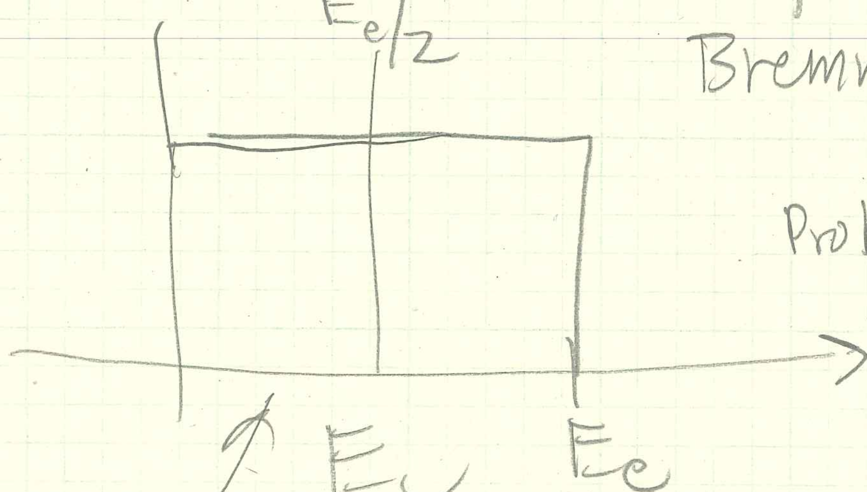
"energy straggling"

Particularly Weird for e^-.

Why?, Look at RPP.

$$e^- + \text{matter} \rightarrow e^- + \gamma + \text{matter}$$

↑ Bremm



$$\text{Prob} \sim \frac{X}{X_0}$$

not gaussian!