Physics 25 Problem Set 6

Harry Nelson

due Monday, May 15

Please make your work neat, clear, and easy to follow. It is hard to grade sloppy work accurately. Generally, make a clear diagram, and label quantities. Derive symbolic answers, and then plug in numbers after a symbolic answer is available.

---

1. In this problem let's count the number of microstates present in the electromagnetic field inside a conducting box of dimensions $L_x \times L_y \times L_z$. We'll ignore quantum mechanics at first, so this gives the so-called 'ultraviolet catastrophe,' which means the total energy seems to blow up as the wavelength of the radiation goes to zero. Imagine one corner of the cavity being at $(x, y, z) = (0, 0, 0)$ and then the cavity being in $0 < x < L_x$, $0 < y < L_y$, $0 < z < L_z$. In this situation, the electric field $\mathbf{E}(x, t)$ has the following form:

$$\begin{aligned} \mathbf{E}(x,t) &= (E_1\mathbf{n}_1 + E_2\mathbf{n}_2) \times f(x,y,z,t) \\ f(x,y,z,t) &= \sin(k_x x)\sin(k_y y)\sin(k_z z)\cos(\omega t) \end{aligned}$$

   In the above, $\mathbf{n}_1$ and $\mathbf{n}_2$ are unit vectors that describe the two directions of $\mathbf{E}$ (polarizations) that are perpendicular to the wavevector $\mathbf{k} = k_x\hat{\boldsymbol{x}} + k_y\hat{\boldsymbol{y}} + k_z\hat{\boldsymbol{z}}$, and E_1 and E_2 are the electric field strengths in those directions. Each of k_x, k_y an k_z must be taken to be positive, because negative values for these wavevector components merely flip the sign of $f(x, y, z, t)$, which is already described by negative values of E_1 and E_2.

   (a) The components of $\mathbf{E}$ must satisfy the wave equation,

   $$\nabla^2\mathbf{E} = \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t^2}$$

   Use this equation to relate $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$ to ω. Then, you know that the wavelength $\lambda = (c/\nu) = (2\pi c/\omega)$, you can relate k to λ.

   (b) Show that:

   $$\Delta k = -2\pi\frac{\Delta\lambda}{\lambda^2}$$

   (c) For $\mathbf{E}$ to vanish at the walls of the cavities, k_x, k_y, and k_z (which are all positive in value) must take on only certain values; each is proportional to a positive integer; we'll call the respective integers n_x, n_y, and n_z. Find the constant of proportionality between k_x and n_x, k_y and n_y, and k_z and n_z.

   (d) To count the number of microstates, it is useful visualize the 3-dimensional space of k_x, k_y, and k_z; this is sometimes called k-space. Since all three of these components are positive, only one octant of 3-d space (that octant where the components are positive) is present in k-space. Imagine a dot in k-space that corresponds to each value of (k_x, k_y, k_z) that is allowed by the boundary conditions. What is the *density* of these dots; that is, in a given volume of k-space, there will be some number of dots; find the ratio of that number of dots to the volume of k-space they inhabit. A simple way to do this is just consider 1 dot and reason out the volume that it sits at the center of. Note that the volume here is not in *physical space*, so the dimensions of the density are not $1/(\text{length})^3$.

(e) Now we count the number of microstates between wavelengths of λ and $\lambda + \Delta\lambda$:

i. Find the volume of k-space that is bounded by the k that corresponds to λ and the $k+\Delta k$ that corresponds to $\lambda + \Delta\lambda$. Evaluate the volume in terms of k, Δk, and constants like π. Draw a picture of this volume... it is a thin sheet of thickness Δk that covers 1/8 portion of a sphere of radius k.

ii. Use the density of dots (that is, density of microstates) and the volume to to calculate the number of microstates between k and $k + \Delta k$. Then convert the k and the Δk to λ and $\Delta\lambda$.

(f) If the walls of the cavity are at absolute temperature T, and if the electromagnetic field is in thermal equilibrium with the walls, then each microstate of the field must have energy $2kT$; where the 2 arises from the 2 orientations of the electric field. Use this concept of thermal equilibrium to deduce that the *density* of energy inside the wavelength interval $\Delta\lambda$, that is, the total energy divided by the volume of the cavity, denoted ρ_E, is:

$$\rho_E = \frac{8\pi kT}{\lambda^4}\,\Delta\lambda$$

.

(g) The *flux* of radiation energy out of a small hole in the cavity will then be equal to the velocity times the energy density; the flux in the wavelength interval $\Delta\lambda$, which the book (page 26) calls $E(\lambda, T)$, is then:

$$E(\lambda, T) = c \times \frac{\rho_E}{\Delta\lambda} = \frac{8\pi ckT}{\lambda^4}$$

Show that this equation agrees with Equation (39a) on page 26 of the book, in the limit that $(hc/(\lambda kT)) << 1$. Note that the quantum-mechanical constant h is absent from this equation.

2. Make a log-log plot of $E(\lambda, T)$ for wavelengths from 10 nm to 10 cm for the three temperatures (on the same plot):

(a) $T = 2.725\,\mathrm{K}$, the temperature of the radiation left over from the Big Bang.

(b) $T = 300\,\mathrm{K}$, which is about room temperature.

(c) $T = 4300\,\mathrm{K}$, which is about the temperature of the sun.

There is a mathematica notebook on the course Web page that should help you with this plot. Use expression (35a) from page 24 of the book to indicate the $\lambda_{\max}$ on the plot for each temperature.
