Electromagnetic Equations

$$\text{div}\,\vec{E} = 4\pi\rho \qquad \text{div}\,\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$\rho = \frac{\text{local charge}}{\text{volume}}$$

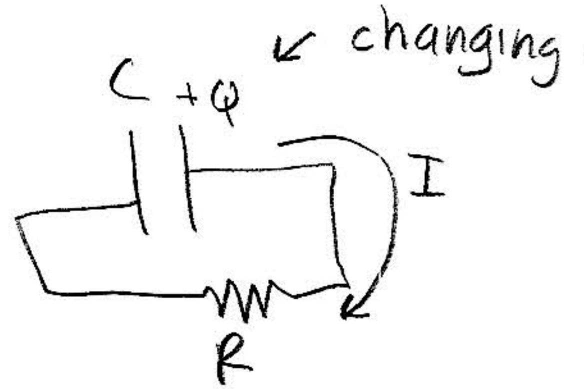
$$\underbrace{\text{div}\,\vec{J} = -\frac{\partial \rho}{\partial t}}_{\text{always true.}} \qquad \vec{J} = \frac{\text{charge}}{\text{area time}} \times (\widehat{\text{direction}})$$

when $\vec{J}$ is constant in time $\left(\frac{\partial \vec{J}}{\partial t} = 0\right)$

$$\text{curl}\,\vec{B} = \frac{4\pi}{c}\vec{J} \qquad \text{curl}\,\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

Contradiction

suppose $\frac{\partial \rho}{\partial t} \neq 0$

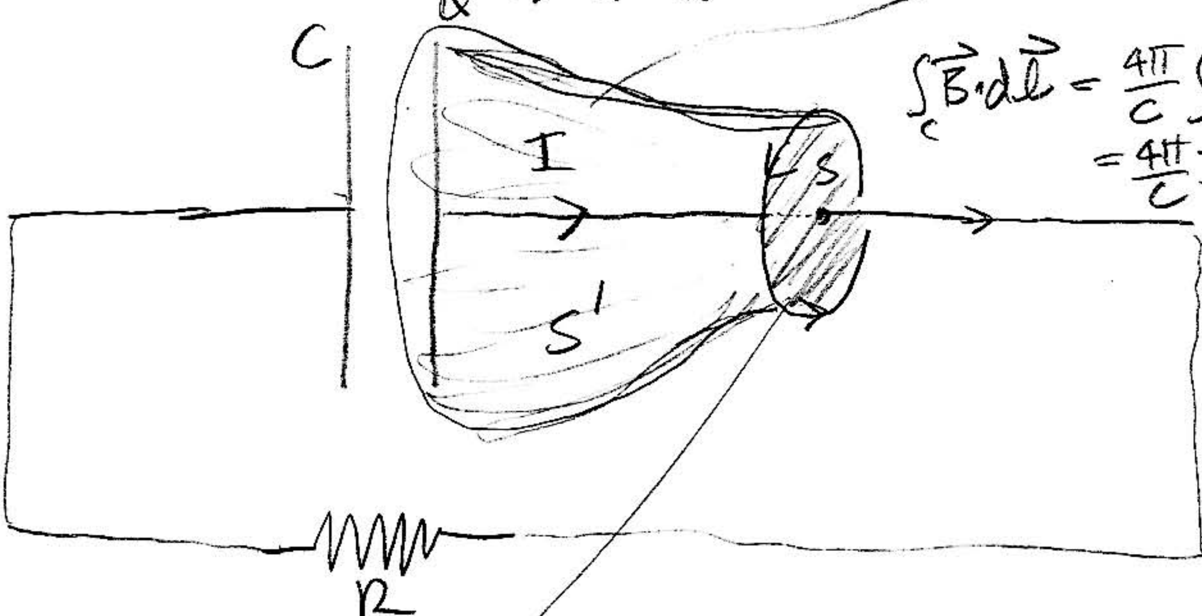


then $\text{div}\,\vec{J} = -\frac{\partial \rho}{\partial t} \neq 0$

but... $\underbrace{\text{div}\left(\text{curl}\,\vec{B}\right)}_{\text{(no sources)}} = 0 = \frac{4\pi}{c}\,\text{div}\,\vec{J}$

That's the contradiction.

Integral Statement



$$\int_C \vec{B} \cdot d\vec{l} = \frac{4\pi}{c} \int_S \vec{J} \cdot d\vec{A} = \frac{4\pi}{c} I$$

but that is not the only surface one could use!

$$\int_{S'} J \cdot dA = 0 \;!$$

so, $\quad \text{curl}\,\vec{B} = \frac{4\pi}{c} \vec{J} + \left(\begin{matrix}\text{new} \\ \text{term}\end{matrix}\right)$

some $\frac{\partial(\;)}{\partial t}$

Look at $\quad \text{curl}\,\vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$

recall symmetry of $\vec{E}, \vec{B}$ in "Lorentz" transformation

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel} \qquad \vec{E}'_{\perp} = \gamma(\vec{E}_{\perp} + \vec{\beta} \times \vec{B}_{\perp})$$
$$\vec{B}'_{\parallel} = \vec{B}_{\parallel} \qquad \vec{B}'_{\perp} = \gamma(\vec{B}_{\perp} - \vec{\beta} \times \vec{E}_{\perp})$$
– sign.

$$\mathrm{curl}\, \vec{B} = \frac{4\pi}{c}\vec{J} + \underbrace{\frac{1}{c}\frac{\partial \vec{E}}{\partial t}}$$

changing $\vec{E}$ field,
integrated across surface,
gives a "circular" $\vec{B}$
(like E.M.F.)
"LENZ" ⇒ other way.

$$\mathrm{curl}\, \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$
$$\mathrm{curl}\, \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J}$$
$$\mathrm{div}\, \vec{E} = 4\pi\rho$$
$$\mathrm{div}\, \vec{B} = 0$$

Maxwell's Equations (CGS)

MKS/SI:

$$\mathrm{curl}\, \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\mathrm{curl}\, \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}$$
$$\mathrm{div}\, \vec{E} = \varepsilon_0 \rho$$
$$\mathrm{div}\, \vec{B} = 0 \qquad \mu_0 \varepsilon_0 = \frac{1}{c^2}$$

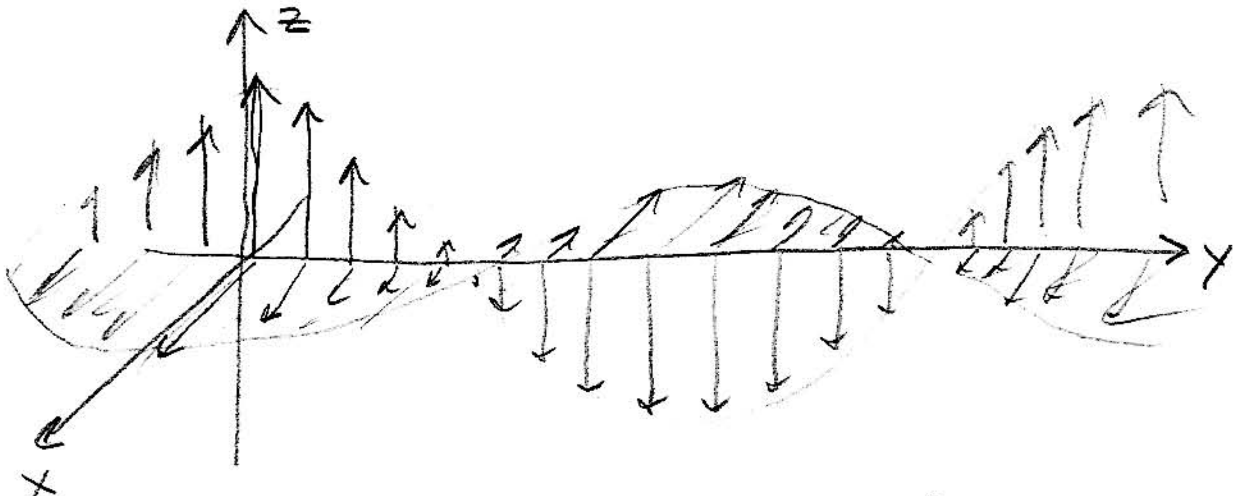
→ And let there be light

$\rho = \vec{J} = 0$ (free space)

$$\text{curl } \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\text{curl } \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

Ansatz: $\vec{E} = \hat{z} E_0 \sin(y - vt)$



$$\vec{B} = \hat{x} B_0 \sin(y - vt)$$

Try now

$$\text{curl } \vec{E}: \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & E_0 \sin(y-vt) \end{vmatrix} = \hat{x} \cdot \frac{\partial}{\partial y}\left(E_0 \sin(y-vt)\right) - \hat{y}\, \cancel{\frac{\partial}{\partial x}\left(E_0 \sin(y-vt)\right)}\, 0$$

$$= \hat{x} E_0 \cos(y - vt)$$

$$\text{curl } \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_0 \sin(y-vt) & 0 & 0 \end{vmatrix} = \begin{array}{l} \hat{x} \cdot 0 \\ -\hat{y} \cdot \left(-\frac{\partial}{\partial z} B_0 \sin(y-vt)\right) \\ + \hat{z}\left(-\frac{\partial}{\partial y} B_0 \sin(y-vt)\right) \end{array} = -\hat{z} B_0 \cos(y-vt)$$

$$-\frac{1}{c}\frac{\partial}{\partial t}\vec{B} = -\frac{1}{c}\frac{\partial}{\partial t}\hat{x}B_0\sin(y-vt)$$

$$= \frac{v}{c}\hat{x}B_0\cos(y-vt)$$

$$\text{curl } \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\hat{x}E_0\cos(y-vt) = \frac{v}{c}\hat{x}B_0\cos(y-vt)$$

$$E_0 = \frac{v}{c}B_0$$

$$\text{curl } \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

$$\frac{\partial \vec{E}}{\partial t} = \frac{\partial}{\partial t}\left(\hat{z}E_0\sin(y-vt)\right) = -v\hat{z}E_0\cos(y-vt)$$

$$-\hat{z}B_0\cos(y-vt) = -\frac{v}{c}\hat{z}E_0\cos(y-vt)$$

$$B_0 = \frac{v}{c}E_0$$

$$\cancel{E_0} = \frac{v}{c}\cdot\frac{v}{c}\cancel{E_0}$$

$$\left(\frac{v}{c}\right)^2 = 1 \qquad \boxed{v = \pm c}$$