Divergence of a Vector Function

Simply put: scalar function $\xrightarrow{\text{(derivatives)}}$ vector function

$$\vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{x} + \frac{\partial \phi}{\partial y}\hat{y} + \frac{\partial \phi}{\partial z}\hat{z}$$

$\hat{i} = \hat{x}$ $\quad \hat{j} = \hat{y}$ $\quad \hat{k} = \hat{z}$

That is the $\boxed{\text{Gradient}}$ of ϕ, which makes a vector function out of a scalar function

How about using derivatives to "do something" to a vector function

$$\text{div}\,\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \quad \text{(cartesian only)}$$

$$= \vec{\nabla}\cdot\vec{F}$$

$$\neq \frac{\partial E_r}{\partial r} + \frac{\partial E_\theta}{\partial \theta} + \frac{\partial E_\phi}{\partial \phi} \quad \text{in spherical! (Important)}$$

not the only way to "get something" from a vector field,

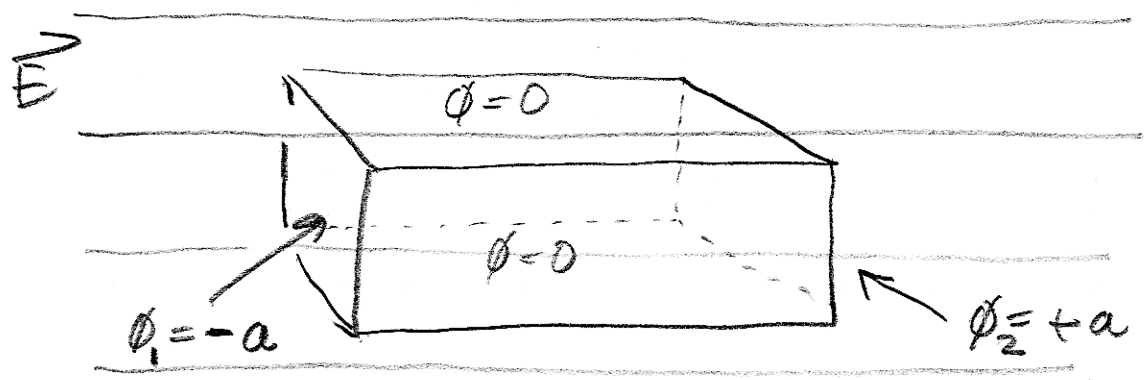
$$\frac{\partial E_x}{\partial y}, \frac{\partial E_x}{\partial z}, \frac{\partial E_y}{\partial x}, \frac{\partial E_y}{\partial z}, \frac{\partial E_z}{\partial x}, \frac{\partial E_z}{\partial y}$$

are not yet involved!

What does div $\vec{E}$ mean?

① look at a constant $\vec{E}$ ... $\vec{\nabla}\cdot\vec{E} = 0$!

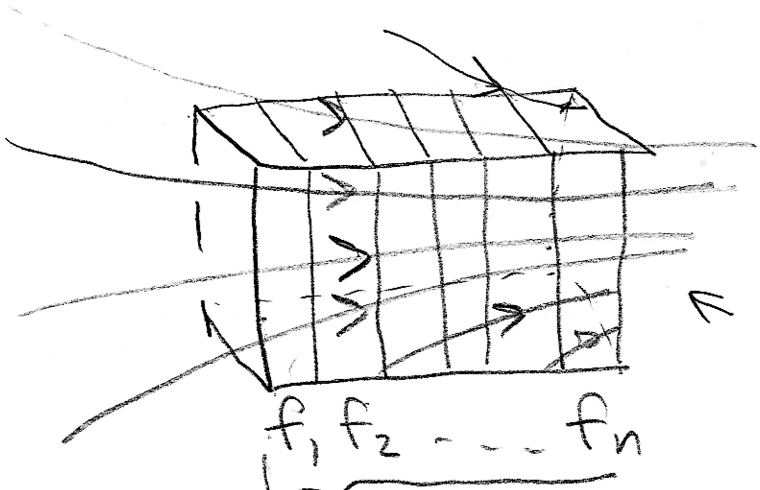
consider flux



net $\phi_1 + \phi_2 = 0$! Orientation of surface unimportant must be closed.

Flux through a closed surface arises from change in $\vec{E}$.

②



more flux!

$\sum f_i = \phi$

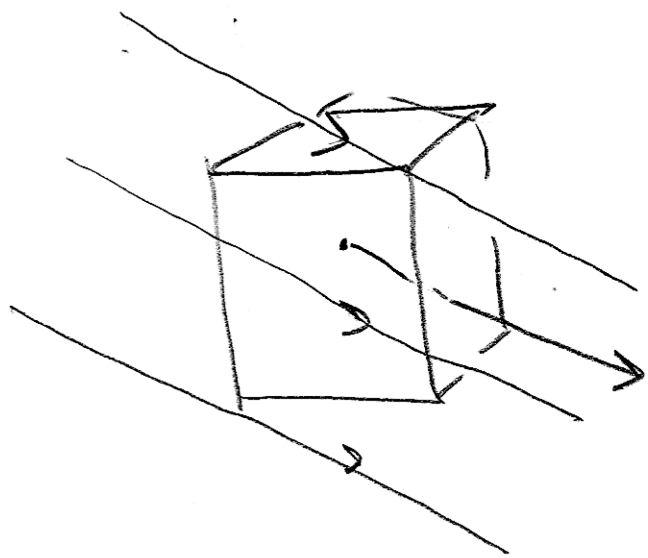
flux gets smaller as volume gets smaller...

$$\vec{\nabla}\cdot\vec{E} = \lim_{V \to 0} \frac{1}{V} \int_{\substack{\text{surface enclosing} \\ V}} \vec{F}\cdot d\vec{a}$$

The flux per unit volume --

Visualization for $\vec{E}$ field ...

can always make V so tiny that $\vec{E}$ looks constant!!



✓ net flux in an infinitesimal arises from a new field line starting

New field line arises from presence of electric charge in box!

$$\underset{\substack{\text{Surface}\\\text{enclosing}\\\text{charge}}}{\int} \vec{E} \cdot d\vec{a} = 4\pi Q_{enclosed} = 4\pi \int_V \rho \, dV$$

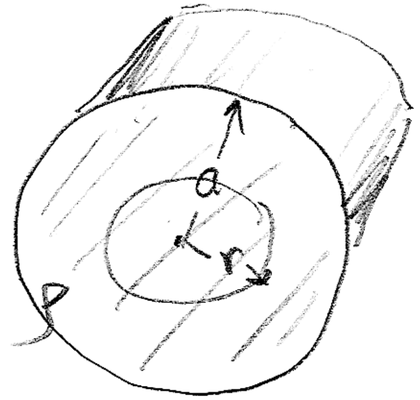
$$= \sum_{i=1}^{N} V_i \left[\frac{\int \vec{E} \cdot d\vec{a}}{V_i} \right]$$

$$= \int_V \vec{\nabla} \cdot \vec{E} \, dV$$

$$\boxed{\vec{\nabla} \cdot \vec{E} = 4\pi\rho}$$

aka, a "Maxwell Equation"

Example: cylinder, ∞ length, filled with constant charge density ρ up to distance a from axis



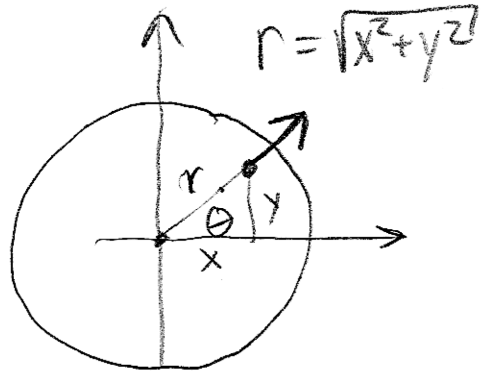
$r<a$:

Gauss: $2\pi r \cdot \cancel{L} \cdot E = 4\pi\rho \cdot \cancel{L} \cdot \pi r^2$

$$\boxed{E = 2\pi\rho r \quad r<a}$$

$$2\pi r \cdot \cancel{L} \cdot E = 4\pi\rho \cdot \cancel{L} \cdot \pi a^2 \quad r>a$$

$$\boxed{E = 2\pi\rho \cdot \frac{a^2}{r}} \quad r>a$$



$$E_x = E_r \cdot \frac{x}{\sqrt{x^2+y^2}}$$

$$E_y = E_r \cdot \frac{y}{\sqrt{x^2+y^2}}$$

$\boxed{r<a}$: $E_x = 2\pi\rho\sqrt{x^2+y^2} \cdot \frac{x}{\sqrt{x^2+y^2}}$

$$E_x = 2\pi\rho x$$

$$E_y = 2\pi\rho y$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 2\pi\rho + 2\pi\rho = 4\pi\rho$$

$\boxed{r>a}$ $E_x = 2\pi a^2 \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{x}{\sqrt{x^2+y^2}} = 2\pi a^2 \frac{x}{x^2+y^2}$

$$E_y = 2\pi a^2 \frac{y}{x^2+y^2}$$

$$\frac{\partial E_x}{\partial x} = 2\pi a^2 \left[\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}\right]$$

$$= 2\pi a^2 \left[\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}\right] = 2\pi a^2 \frac{y^2-x^2}{x^2+y^2}$$

$$\frac{\partial E_y}{\partial y} = 2\pi a^2 \left[\frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}\right]$$

$$= 2\pi a^2 \left[\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right] = -2\pi a^2 \frac{y^2-x^2}{x^2+y^2}$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0$$

Laplacian:

$$\vec{E} = -\vec{\nabla}\phi = -\left(\hat{x}\frac{\partial\phi}{\partial x} + \hat{y}\frac{\partial\phi}{\partial y} + \hat{z}\frac{\partial\phi}{\partial z}\right)$$

$$\vec{\nabla}\cdot\vec{E} = -\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right)$$

$$= -\underbrace{\nabla^2}_{\text{"Laplacian"}}\phi$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = -4\pi\rho$$