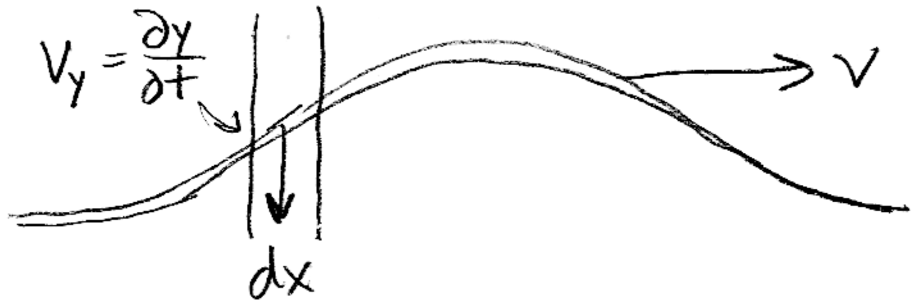
Energy in Waves

Transverse waves on string

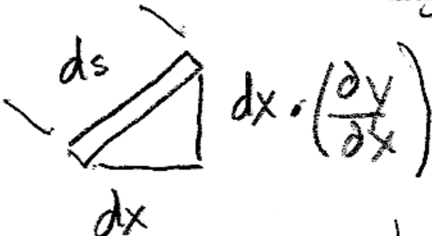


Kinetic Energy: $dK = \frac{1}{2}\underbrace{(\mu\, dx)}_{\text{mass}} V_y^2$

μ = linear mass density

$$\frac{dK}{dx} = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2$$

Potential Energy



$$ds = \sqrt{(dx)^2 + \left(\frac{\partial y}{\partial x}\right)(dx)^2}$$

change in length

$$ds - dx = \underbrace{dx\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2}} - dx$$

$$= dx\left(1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right) - dx$$

$$ds - dx = \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2$$

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Work done to lengthen the string

$$dU = T\,(ds - dx)$$

$$= \frac{1}{2} T \left(\frac{\partial y}{\partial x}\right)^2 dx \quad [\text{second order}]$$

$$\frac{dU}{dx} = \frac{1}{2} T \left(\frac{\partial y}{\partial x}\right)^2$$

Total Energy Density

$$\frac{dE}{dx} = \frac{1}{2} T \left(\frac{\partial y}{\partial x}\right)^2 + \frac{1}{2}\mu \left(\frac{\partial y}{\partial t}\right)^2$$

Pulse $\quad y(x,t) = f(x \pm vt)$

$$\frac{\partial y}{\partial x} = f'(x \pm vt) \qquad \frac{\partial y}{\partial t} = \pm v f'(x \pm vt)$$

$$\frac{dE}{dx} = \frac{1}{2}\left(T + \mu v^2\right)\left(f'(x \pm vt)\right)^2$$

but, $v^2 = \dfrac{T}{\mu}$

$$\frac{dE}{dx} = T\left(f'(x \pm vt)\right)^2 = \mu v^2 \left(f'(x \pm vt)\right)^2$$

$$f(x,t) = A \sin\left(\frac{2\pi}{\lambda}(x - vt)\right)$$

$$= A \sin(kx - \omega t)$$

NOTE

$$vT = \lambda \qquad \omega = 2\pi\nu$$

$$\frac{1}{T} = \nu = \frac{v}{\lambda} \qquad \omega = \frac{2\pi v}{\lambda} = kv$$

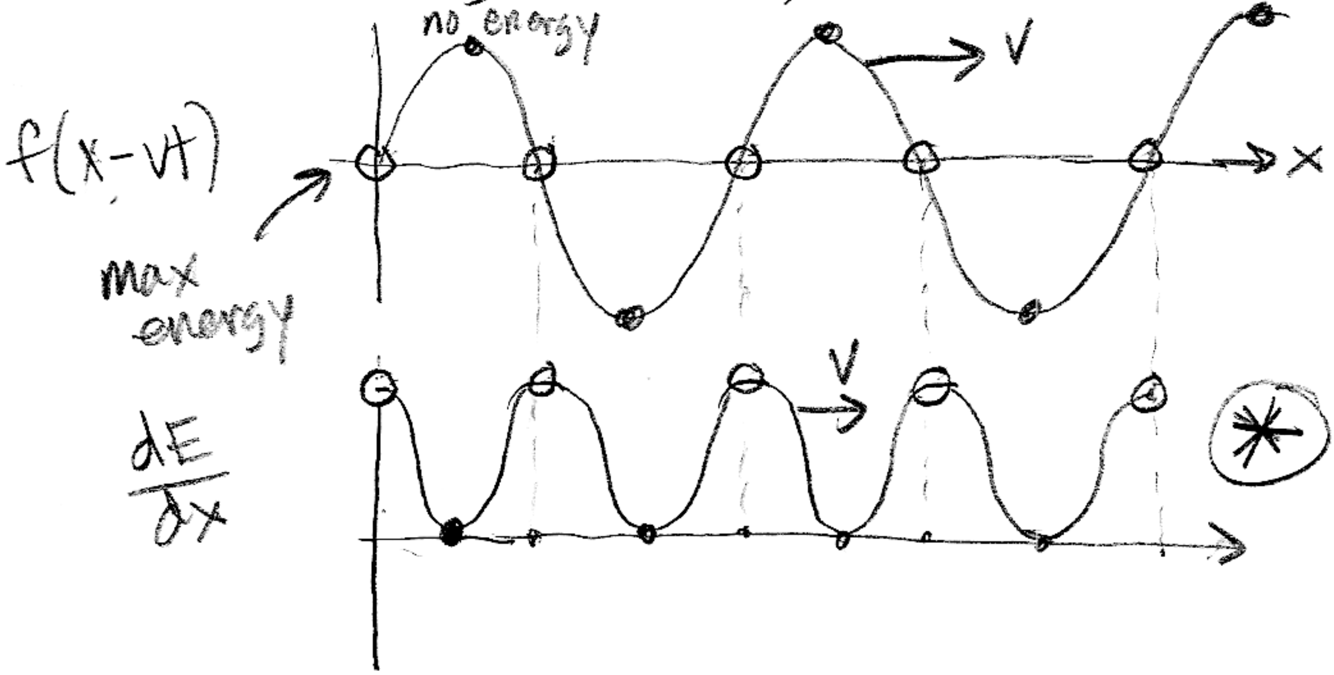
$$f'(x-vt) = kA\cos(kx-\omega t)$$

$$\frac{dE}{dx} = \mu \underbrace{v^2 \cdot k^2}_{\omega^2} \cdot A^2 \cos^2(kx-\omega t)$$

Total Energy Density $\boxed{\frac{dE}{dx} = \mu\omega^2 A^2 \cos^2(kx-\omega t)}$

- note proportionalities, power laws

$\frac{dE}{dx} \propto \mu$, $\propto \omega^2$, $\propto A^2$



- not on string ... longitudinal wave in bar, air... still

$$\frac{dE}{dx} = \mu\omega^2 A^2 \cos^2(kx-\omega t)$$

change to ρ $\quad \omega = kv$ still $\quad v = \sqrt{\frac{Y}{\rho}}, \sqrt{\frac{\gamma RT}{A_w}}$

Sum energy in the sinusoidal wave (refer back to * graph)

- Freeze time at some t

- $$\bar{E}_\lambda = \int_x^{x+\lambda} \left(\frac{dE}{dx}\right) dx = \mu\omega^2 A^2 \int_x^{x+\lambda} dx \cos^2(kx - \phi)$$

$$\phi = \omega t$$

$$\xi = kx - \phi$$

$$d\xi = k\,dx \qquad k\cdot\lambda = \frac{2\pi}{\lambda}\cdot\lambda = 2\pi$$

$$\bar{E}_\lambda = \int_x^{x+\lambda} \left(\frac{dE}{dx}\right) dx = \frac{\mu\omega^2 A^2}{k} \int_{kx-\phi}^{kx-\phi+2\pi} d\xi \cos^2\xi$$

$$\cos 2\xi = \cos^2\xi - \sin^2\xi = \cos^2\xi - (1 - \cos^2\xi)$$

$$= 2\cos^2\xi - 1$$

$$\cos^2\xi = \frac{1}{2}(1 + \cos 2\xi)$$

$$\int d\xi \cos^2\xi = \frac{1}{2}\xi + \frac{1}{4}\sin 2\xi$$

$$\int_{kx-\phi}^{kx-\phi+2\pi} d\xi \cos^2\xi = \frac{1}{2}\xi \Big|_{kx-\phi}^{kx-\phi+2\pi} + \frac{1}{4}\sin 2\xi \Big|_{kx-\phi}^{kx-\phi+2\pi} \rightarrow 0$$

$$= \frac{1}{2} \times 2\pi$$

$$\bar{E}_\lambda = \int_x^{x+\lambda} \left(\frac{dE}{dx}\right) dx = \frac{\mu\omega^2 A^2}{\left(\frac{\cancel{2\pi}}{\lambda}\right)} \cdot \frac{1}{2} \times \cancel{2\pi}$$

$$= \underbrace{\frac{1}{2}\mu\omega^2 A^2}_{\frac{1}{2}\text{ the peak energy}} \cdot \underbrace{\lambda}_{\text{over one wavelength}}$$

<u>note</u>: $y = A\cos(kx - \omega t)$

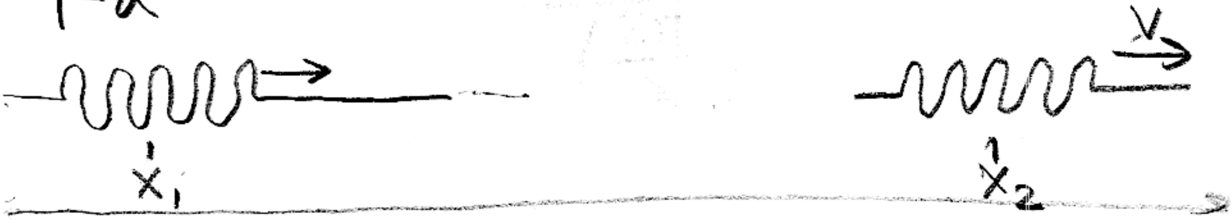
$$\frac{\partial y}{\partial t} = \underbrace{\omega A}_{\text{maximum velocity } v_0} \sin(kx - \omega t)$$

$$\boxed{\bar{E}_\lambda = \frac{1}{2} \underbrace{(\mu \cdot \lambda)}_{\text{mass in one }\lambda} \underbrace{v_0^2}_{\text{peak velocity}^2}}$$

Into the second, third dimensions

Focus on energy -- know it is conserved, but in 2,3 dimensions the density of energy decreases

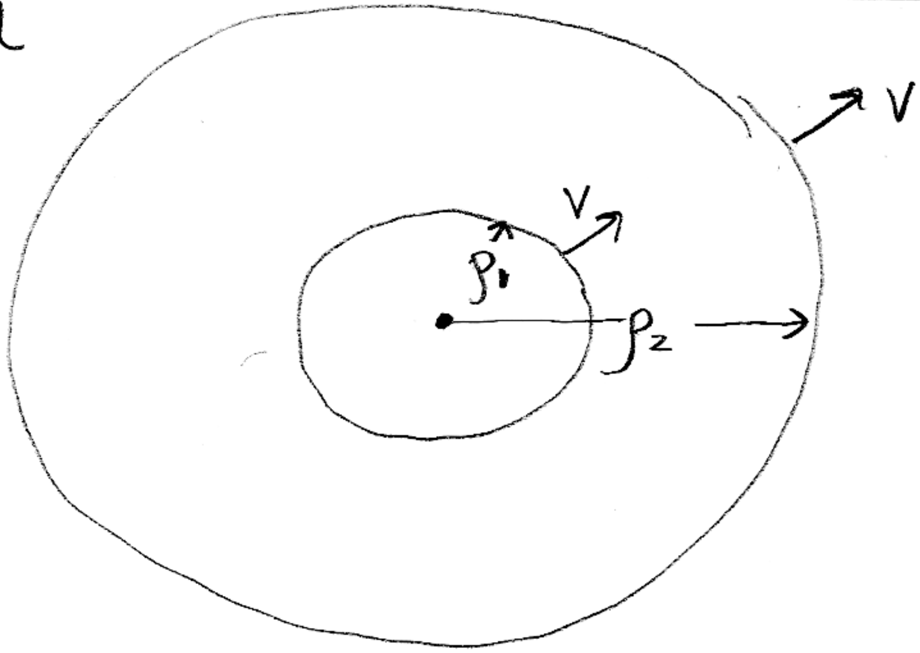
1-d



no dissipation

2-d

look at wave from above



speed doesn't change!

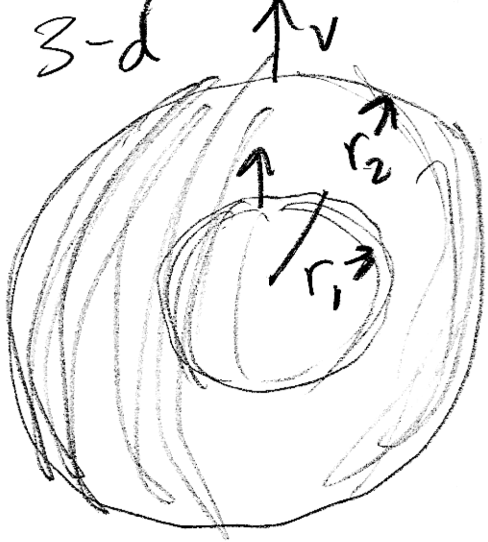
$$\underbrace{2\pi\rho_1}_{\text{circumference}} \cdot \underbrace{\mathcal{E}_1}_{\frac{\text{energy}}{\text{length}}} = 2\pi\rho_2\,\mathcal{E}_2$$

$$\mathcal{E}_2 = \left(\frac{\rho_1}{\rho_2}\right)\mathcal{E}_1$$

$\mathcal{E}_2 \propto A_2^{\,2}$ ← wave amplitude at ρ_2

$$\boxed{A_2 \propto \sqrt{\frac{\rho_1}{\rho_2}}\,A_1}$$

3-d



$$4\pi r_1^2 \cdot \sigma_1 = 4\pi r_2^2 \cdot \sigma_2$$

surface area

$\frac{\text{energy}}{\text{area}}$

$$\sigma_2 = \left(\frac{r_1}{r_2}\right)^2 \sigma_1$$

$$A_2 \propto \left(\frac{r_1}{r_2}\right) A_1$$

n-dimensions

$$A_2 \propto \left(\frac{r_1}{r_2}\right)^{\frac{(n-1)}{2}} A_1$$