## Fourier Series

Cannot "pluck" a string in a sinusoidal pattern:

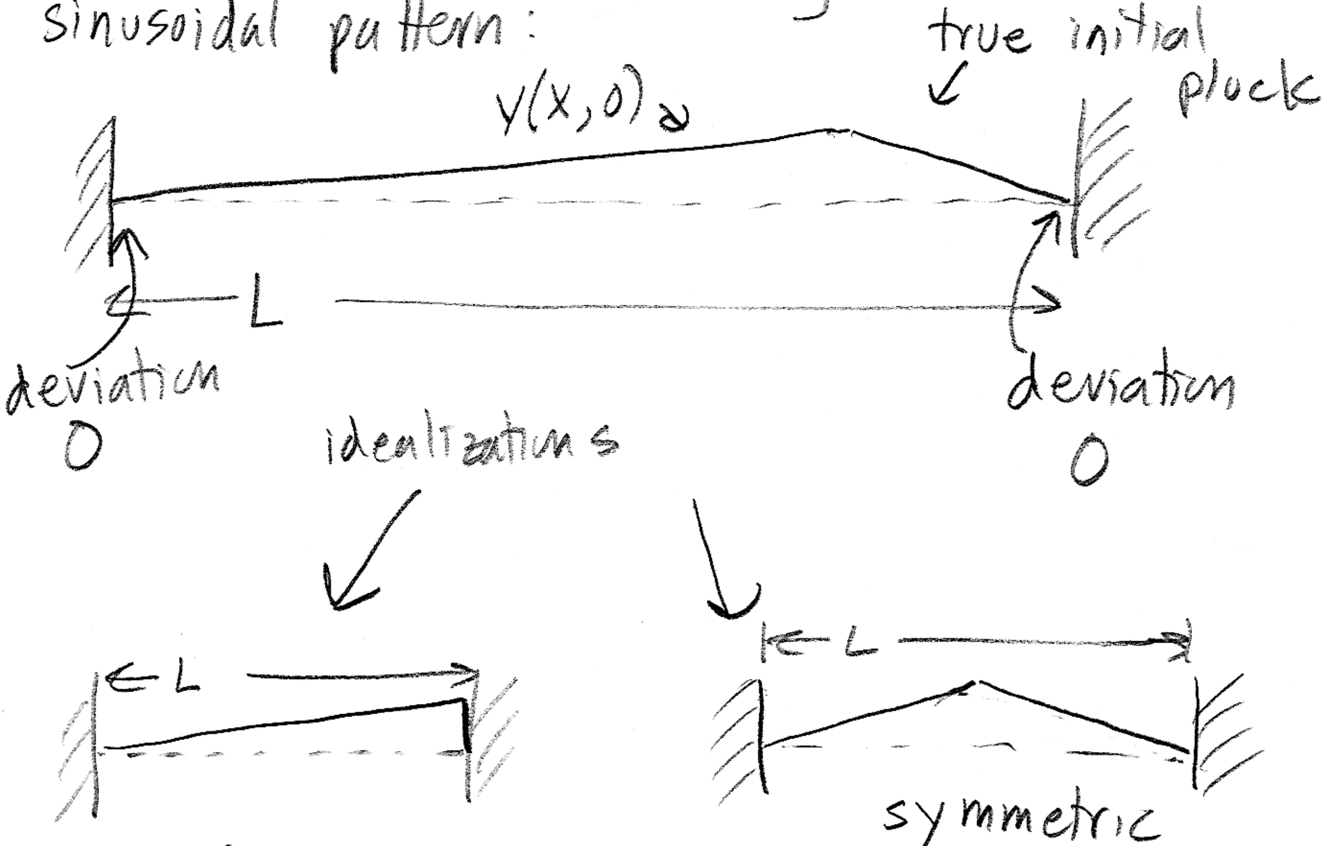


Fig 6-16 in book.

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$ Fourier Series

then

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) \underline{\cos(\omega_n t)}$$

normal modes oscillate in time with

$$k_n = \frac{n\pi}{L} = \frac{2\pi}{\left(\frac{2L}{n}\right)} = \frac{2\pi}{\lambda_n}$$

$\omega_n = v \cdot k_n \qquad v = \sqrt{\frac{T}{\mu}}$

why cos? ($t=0$ condition)

How to find $B_n$ ....

① multiply both sides by
$\sin\left(\frac{n_1\pi}{L}x\right) \rightarrow n_1 =$ arbitrary integer $\geq 1$

② Integrate between $0$ & $L$

$$\int_0^L dx\, y(x,0) \cdot \sin\left(\frac{n_1\pi}{L}x\right)$$

$$= \sum_{n=1}^{\infty} B_n \underbrace{\int_0^L dx \sin\left(\frac{n_1\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)}_{\text{study this integral}}$$

take $\xi \equiv \frac{\pi}{L}x \quad x=L,\ \xi=\pi$

zeta

$$= \frac{L}{\pi}\int_0^\pi d\xi \sin(n_1\xi)\sin(n\xi)$$

$$\frac{1}{2i}\left(e^{in_1\xi} - e^{-in_1\xi}\right)\frac{1}{2i}\left(e^{in\xi} - e^{-in\xi}\right)$$

$$= \frac{-L}{4\pi}\int_0^\pi \left[d\xi\left(e^{i(n_1+n)\xi} - \underbrace{2\mathrm{Re}\left(e^{i(n_1-n)\xi}\right)}_{z+z^*} + e^{-i(n_1+n)\xi}\right)\right]$$

$$\int_0^\pi d\xi\, e^{\pm i(n_1+n)\xi} = \frac{1}{\pm i(n_1+n)}\, e^{\pm i(n_1+n)\xi}\Big|_0^\pi$$

$$= \frac{\pm 1}{i(n_1+n)}\left(e^{\pm i(n_1+n)\pi} - 1\right)$$

Case A: $n_1+n =$ even #

$$e^{\pm i(n_1+n)\pi} = 1$$

integral $= 0$

Case B: $n_1+n =$ odd #

$$e^{\pm(n_1+n)\pi} = -1$$

integral $= \dfrac{\mp 1}{i(n_1+n)}$

$$\int_0^\pi d\xi\left(e^{i(n_1+n)\xi} + e^{-i(n_1+n)\xi}\right) = 0$$

$$\int dx \sin\left(\frac{n_1\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right) = \frac{L}{2\pi}\mathrm{Re}\left[\int_0^\pi d\xi\, e^{i(n_1-n)\xi}\right]$$

$n_1 \neq n$

$$= \frac{L}{2\pi}\mathrm{Re}\left[\frac{1}{i(n_1-n)}\, e^{i(n_1-n)\xi}\Big|_0^\pi\right] = \frac{L}{2\pi}\mathrm{Re}\left[\frac{\mp 1 - 1}{i(n_1-n)}\right]$$

$n_1-n=$ odd / even

$$\int dx \sin\left(\frac{n_1 \pi}{L} x\right) \sin\left(\frac{n\pi}{L} x\right) = 0 \quad \text{when } n_1 \neq n$$

when $n_1 = n$

$$\int dx \sin\left(\frac{n_1 \pi}{L} x\right) \sin\left(\frac{n\pi}{L} x\right) = \frac{L}{2\pi} \int_0^{\pi} d\xi = \frac{L}{2}$$

$$\boxed{\int dx \sin\left(\frac{n_1 \pi}{L} x\right) \sin\left(\frac{n\pi}{L} x\right) = \frac{L}{2} \delta_{n_1 n}}$$

$$\delta_{n_1 n} = 1 \quad \text{when } n_1 = n$$
$$= 0 \quad \text{when } n_1 \neq n$$

$$\int_0^L dx\, y(x,0) \sin\left(\frac{n_1 \pi}{L}\right) dx = \sum_{n=1}^{\infty} B_n \frac{L}{2} \delta_{n_1 n}$$

$$= \frac{L}{2} B_{n_1}$$

or $$\boxed{B_n = \frac{2}{L} \int_0^L dx\, y(x,0) \sin\left(\frac{n\pi}{L} x\right)}$$