Capacitors (Chapter 25)

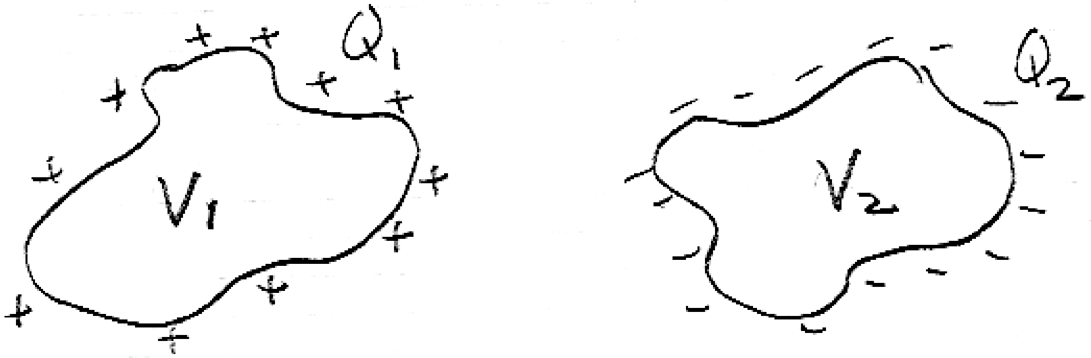
1. Basic Idea
2. Series + Parallel
3. Energy
4. Dielectrics + adaptation of Gauss

Idea

Point charge: $V = \frac{Q}{4\pi\epsilon_0 r}$

Generally, $V \propto Q$ (linear)

Capacitors: equipotentials.



linear $$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}}_{\text{capacitance matrix}} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} + \beta$$

β must be zero, assuming no other fields

What matters most? $V_2 - V_1$

Change varibles: $\Delta V \equiv V_2 - V_1$ $\quad \bar{V} = \frac{1}{2}(V_1 + V_2)$

aka $\begin{pmatrix} \Delta V \\ V \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} \Delta V \\ V \end{pmatrix}$$

inverse of $\begin{pmatrix} -1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

rearrange...

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

convention

$$\frac{1}{2} \begin{pmatrix} -1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \underbrace{\frac{1}{2} \begin{pmatrix} -1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & 1 \end{pmatrix}} \begin{pmatrix} -1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} \Delta Q \\ \frac{1}{2} \bar{Q} \end{pmatrix} = \begin{pmatrix} \tilde{\gamma}_{11} & \cancel{\tilde{\gamma}_{12}} \to 0 \\ 0 \leftarrow \cancel{\tilde{\gamma}_{21}} & \tilde{\gamma}_{22} \end{pmatrix} \begin{pmatrix} \Delta V \\ V \end{pmatrix}$$

argument: ΔQ independent of V

$\bar{Q}$ independent of ΔV

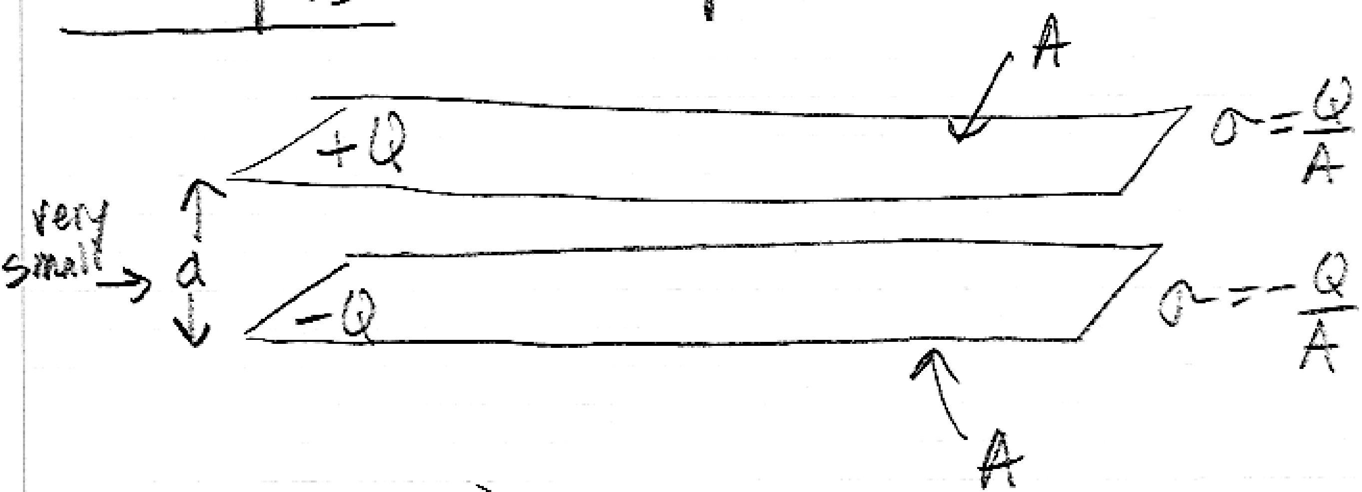
voltage far away depends only on $\bar{Q}$, not ΔQ

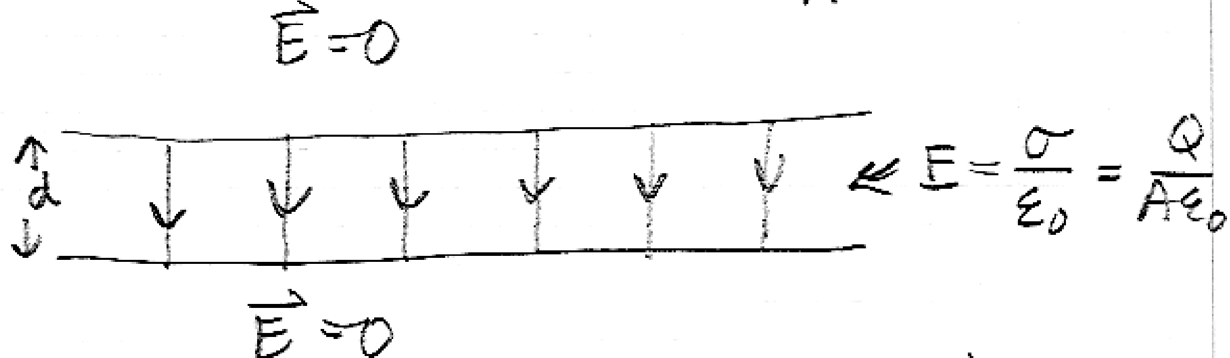
- charge placed out here influences only V, not ΔV

so,
$$\begin{pmatrix} \frac{1}{2}\Delta Q \\ \frac{1}{2}\overline{Q} \end{pmatrix} = \begin{pmatrix} \tilde{\gamma}_1 & 0 \\ 0 & \tilde{\gamma}_2 \end{pmatrix} \begin{pmatrix} \Delta V \\ V \end{pmatrix}$$

$$\frac{1}{2}\Delta Q = C \Delta V \qquad C \equiv \tilde{\gamma}_1 \text{ is "the capacitance"}$$

<u>Examples</u>: Two plates:





$$\Delta V = Ed = \frac{Q}{A\varepsilon_0} \cdot d = \left(\frac{d}{A\varepsilon_0}\right) \cdot Q$$

or
$$Q = \left(\varepsilon_0 \cdot \frac{A}{d}\right) \Delta V = C V_{ab}$$

$$\frac{1}{2}\Delta Q = \frac{1}{2}(Q - (-Q)) = \frac{1}{2} \times 2Q = Q$$

So:
$$\boxed{C = \varepsilon_0 \cdot \frac{A}{d}}$$

Units : $[C] = \left[\frac{Q}{V}\right] = \frac{\text{Coulombs}}{\text{Volt}}$

$\equiv$ Farad

note: $\frac{1}{4\pi\varepsilon_0} = 9.0 \cdot 10^9$

$\varepsilon_0 = \frac{1}{4\pi 9 \cdot 10^9} = 8.85 \cdot 10^{-12} \frac{\text{Farad}}{\text{meter}}$

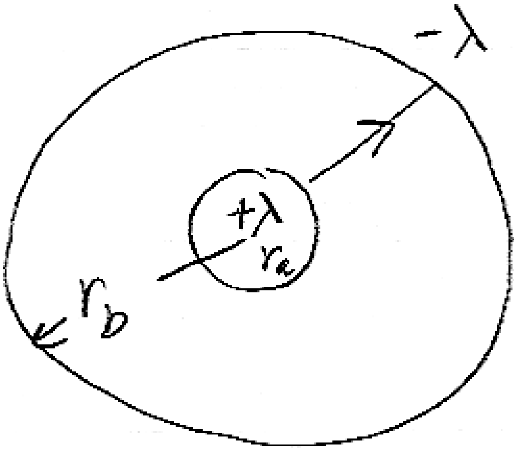
Suppose $d = 10^{-3}$ m, want a 1 Farad capacitor:

$$1 = \varepsilon_0 \frac{A}{d} = \left(8.85 \cdot 10^{-12} \frac{F}{m}\right)\left(\frac{A}{10^{-3} m}\right)$$

$$A = \frac{1}{8.85 \cdot 10^{-9}} m^2 = 1.1 \cdot 10^8 \, m^2$$

If square, $A = L^2$, $\boxed{L \cong \sqrt{10^8 m^2} \simeq 10^4 m \quad L \simeq 10 \text{ km !}}$

Cylindrical



$$E_\rho = \frac{\lambda}{2\pi\varepsilon_0 \rho}$$

$$\int_{r_a}^{r_b} E_\rho \, d\rho = \frac{\lambda}{2\pi\varepsilon_0} \int_{r_a}^{r_b} \frac{d\rho}{\rho}$$

$$V_a - V_b = \Delta V = \frac{\lambda}{2\pi\varepsilon_0} \ln(r_b/r_a)$$

$$\frac{1}{2}\Delta Q = \frac{1}{2} \times L \times (\lambda - (-\lambda))$$

$$= L\lambda = Q$$

$$\lambda = Q/L$$

$$\Delta V = \frac{Q}{2\pi\varepsilon_0} \frac{\ln(r_b/r_a)}{L}$$

$$Q = \underbrace{\left(\frac{2\pi\varepsilon_0}{\ln(r_b/r_a)} \frac{1}{L} \right)}_{C} \cdot \Delta V$$

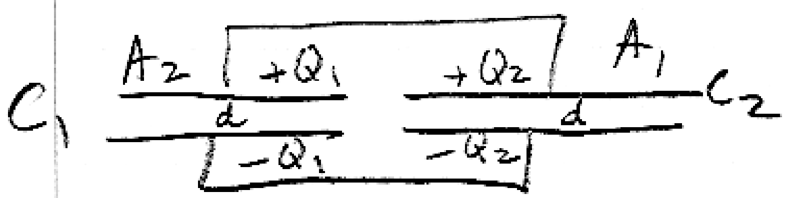
C ; sometimes (like coaxial cable) $\frac{2\pi\varepsilon_0}{\ln(r_b/r_a)}$ is capacitance per unit length.

Series + Parallel

capacitor symbol

Visual: (Parallel)

like A increases



more charge

mnemonic: $C_1 = \varepsilon_0 \frac{A_1}{d}$ $\quad C_2 = \varepsilon_0 \frac{A_2}{d}$

$$C = \frac{\varepsilon_0}{d}(A_1 + A_2) = C_1 + C_2$$

more general:

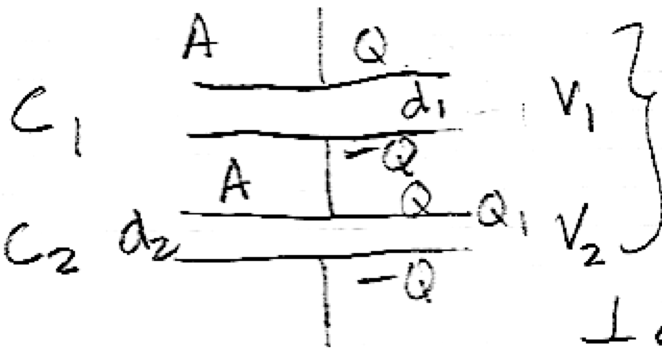
$$\Delta V \ni V = \underline{\underline{\text{same}}}$$

$$Q_1 = C_1 V \qquad Q_2 = C_2 V$$

so $Q_1 + Q_2 = \underline{(C_1 + C_2)}\, V = \underline{C}\, V$

$C_1 + C_2 = C$

Visual: Series



like $\underline{d} = d_1 + d_2$

$$C_1 = \varepsilon_0 \frac{A}{d_1} \qquad C_2 = \varepsilon_0 \frac{A}{d_2}$$

$$\frac{1}{C} \propto d_1 + d_2 \propto \frac{1}{C_1} + \frac{1}{C_2}$$

more formally

$$Q = C_1 V_1 \qquad Q = C_2 V_2$$

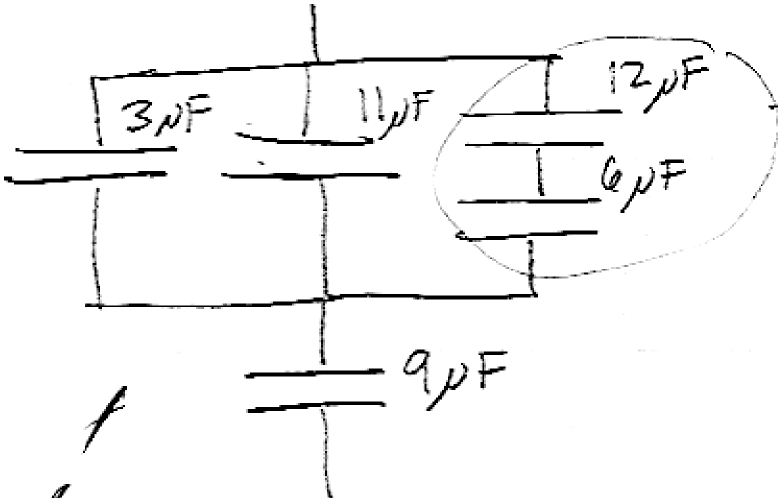
$$V = V_1 + V_2 = \frac{Q}{C_1} + \frac{Q}{C_2} = \left(\frac{1}{C_1} + \frac{1}{C_2}\right) Q$$

$$Q = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}} V = \frac{C_1 C_2}{C_1 + C_2} V$$

so $\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} \qquad C = \frac{C_1 C_2}{C_1 + C_2}$

In this case, C is smaller than either C_1 or C_2, since $1/C$ is larger than either $1/C_1$ or $1/C_2$.

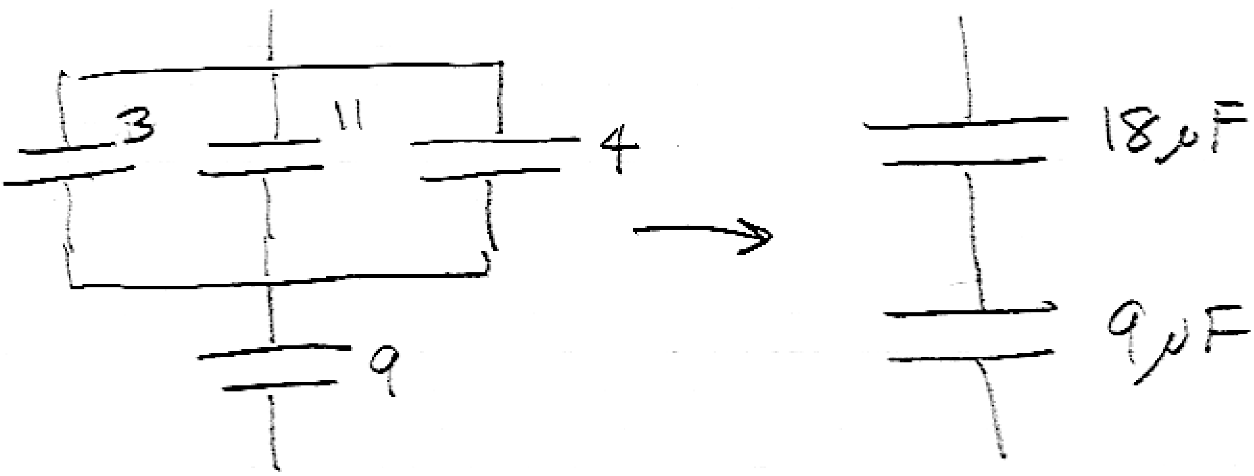
Networks: Successively Reduce them.



$$\frac{1}{12}+\frac{1}{6}=\frac{1}{12}+\frac{2}{12}$$

$$=\frac{3}{12}=\frac{1}{4}$$

like 4μF



$$\frac{1}{18}+\frac{1}{9}=\frac{1}{18}+\frac{2}{18}$$

$$=\frac{3}{18}=\frac{1}{6}$$

like 6μF

equivalent to