Physics 23 Problem Set 7

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Due Monday, November 12

Please make your work neat, clear, and easy to follow. It is hard to grade sloppy work accurately. Generally, make a clear diagram, and label quantities. Derive symbolic answers, and then plug in numbers after a symbolic answer is available.

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1. Let's learn in this problem a bit of the mathematics pertinent to the Maxwell-Boltzmann distribution. A new, special function is useful for this topic, called the *incomplete gamma function*, $\Gamma(a,x)$:

$$\Gamma(a,x) = \int_x^\infty t^{a-1} e^{-t} dt.$$

The *complete* gamma function is $\Gamma(a) = \Gamma(a,0)$.

(a) Evaluate $\Gamma(1)$ (this should be easy!). Take my word for it that $\Gamma(1/2) = \sqrt{\pi}$.

(b) Show, with integration by parts, that:

$$\Gamma(a+1) = a\Gamma(a)$$

and evaluate $\Gamma(2)$ and $\Gamma(3/2)$.

(c) The Maxwell-Boltzmann distribution is given in Equation 18.32 of your text, and is:

$$f(v) = 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 e^{-\frac{mv^2}{2kT}}$$

i. Use calculus to evaluate the most probable speed v_{mp}, and show that:

$$v_{\mathrm{mp}} = \sqrt{\frac{2kT}{m}}$$

ii. Show that you can rewrite $f(v)$ as:

$$f(v) = \frac{4}{\sqrt{\pi}} \frac{1}{v_{\mathrm{mp}}} \left(\frac{v}{v_{\mathrm{mp}}}\right)^2 e^{-\left(\frac{v}{v_{\mathrm{mp}}}\right)^2}$$

iii. Show that the probability that a molecule has a speed greater than a specified velocity $\tilde{v}$, $P(v > \tilde{v})$ is:

$$P(v > \tilde{v}) = \int_{\tilde{v}}^\infty f(v) dv = \frac{\Gamma(3/2, (\tilde{v}/v_{\mathrm{mp}})^2)}{\Gamma(3/2)}$$

(d) Numerically evaluate v_{mp} for:

i. A hydrogen atom on the Sun, where the temperature is 5800 K.

ii. A hydrogen molecule on the Mars, where the temperature is 0° C.

(e) Numerically evaluate the escape velocity for:

   i. The Sun, which has a mass of 2.0×10^{30} kg and a radius of 7.0×10^{8} m. Newton's gravitational constant is $G = 6.67 \times 10^{-11}\,\mathrm{Nm^2kg^{-2}}$.
   ii. Mars, which has a mass of 6.4×10^{23} kg and a radius of 3.4×10^{6} m.

(f) Numerical evaluation of the incomplete gamma function is available at the web page: `http://functions.wolfram.com/webMathematica/FunctionEvaluation.jsp?name=Gamma2`

   i. Numerically evaluate the probability that a hydrogen atom on the Sun in thermal equilibrium has a velocity that exceeds escape velocity.
   ii. Repeat for a hydrogen molecule on Mars.

2. (a) Numerically evaluate $1/kT$ at room temperature ($T = 20^\circ$ C) and at the temperature of the sun ($T = 5800\ K$) in units of (electron volts)$^{-1}$. One electron volt equals 1.6×10^{-19} eV.

   We can simplify a light-emitting diode to a system with two quantum states for an electron trapped in the diode: a ground state and an excited state an energy ΔE above the ground state. The electron in the diode is in thermal equilibrium with the environment at its temperature T Use the Boltzmann factor to numerically evaluate the probability that the electron is in the excited state first at room temperature, and second at the temperature of the sun, for:

   (b) A light emitting diode where $\Delta E = 2.5$ electron-volts, which emits blue light.

   (c) A light emitting diode where $\Delta E = 0.083$ electron-volts, which emits infrared light.

3. 19.40

4. 19.48

5. 19.64