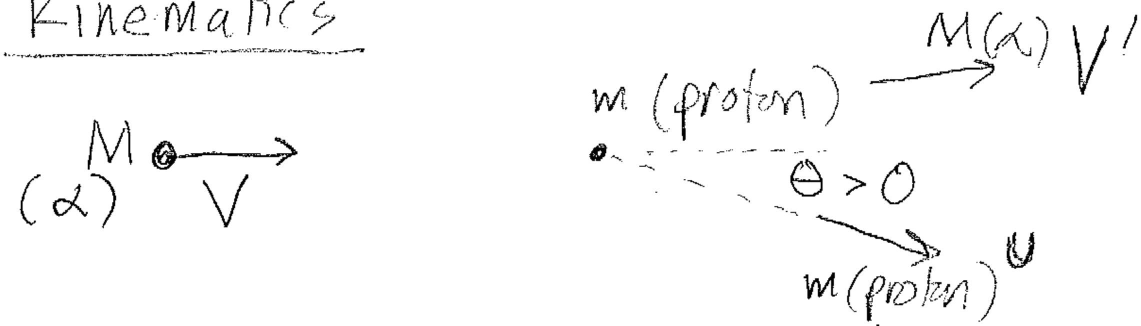
1. Kinematics



Given θ, momentum transfer

$$q = 2\mu V \cos\theta \qquad \mu = \frac{mM}{m+M}$$
(reduced mass)

$$= 2(\mu c)\left(\frac{V}{c}\right)\cos\theta$$

$$\boxed{q = 2\mu c\,\beta\cos\theta}$$

Two ways to derive

(1) Directly from lab kinematics...

$$\frac{1}{2}MV^2 = \frac{1}{2}mv^2 + \frac{1}{2}MV'^2 \rightarrow V^2 - V'^2 = \frac{m}{M}v^2$$

$$M\vec{V} = m\vec{v} + M\vec{V}'$$

$$M\vec{V} - m\vec{v} = M\vec{V}'$$

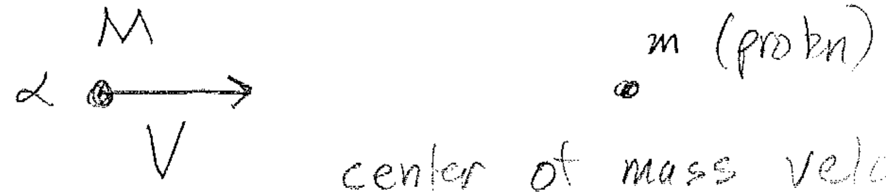
$$M^2V^2 - 2mM\vec{v}\cdot\vec{V} + m^2v^2 = M^2V'^2$$

$$\underbrace{M^2(V^2 - V'^2)}_{\frac{m}{M}v^2} + m^2 v^2 = 2mM\vec{v}\cdot\vec{V}$$

$$(M+m)v^2 = 2M\vec{v}\cdot\vec{V} = 2MvV\cos\theta$$

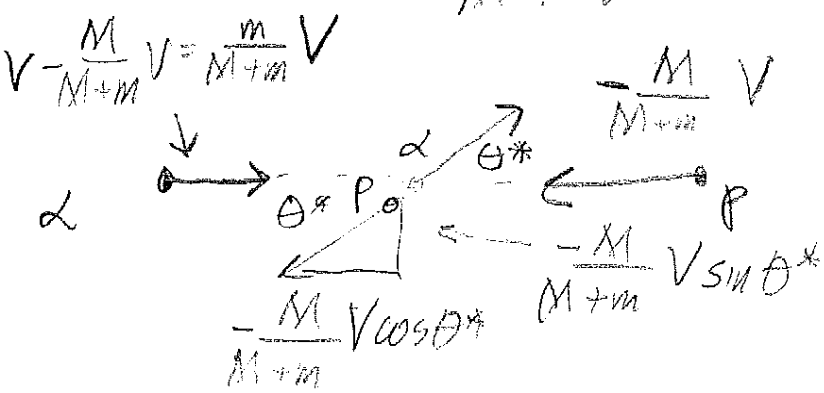
$$mv = 2\frac{mM}{m+M}V\cos\theta = 2\mu V\cos\theta$$

(2) Go back into CM frame, get a little more info

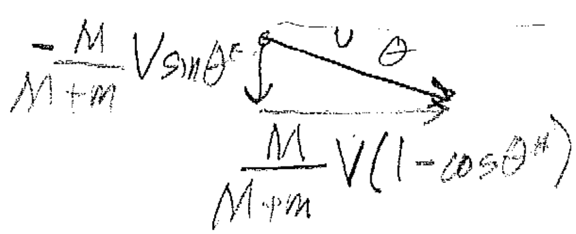


center of mass velocity is

$$\frac{MV}{M+m}$$



Boost proton back to lab



$$v^2 = \left(\frac{M}{M+m}\right)^2 V^2 \underbrace{\left(\sin^2\theta^* + 1 - 2\cos\theta^* + \cos^2\theta^*\right)}_{2(1-\cos\theta^*)}$$

$$v^2 = 2\left(\frac{M}{M+m}\right)^2 V^2 (1-\cos\theta^*)$$

$$v = \sqrt{2}\, \frac{M}{M+m} V \sqrt{1-\cos\theta^*}$$

note $\cos\theta = \dfrac{\frac{M}{M+m} V (1-\cos\theta^*)}{\sqrt{2}\, \frac{M}{M+m} V \sqrt{1-\cos\theta^*}} = \frac{1}{\sqrt{2}} \sqrt{1-\cos\theta^*}$

so $\sqrt{1-\cos\theta^*} = \sqrt{2}\cos\theta$ (actually interesting)

$$\boxed{\begin{aligned} &\text{and} \\ &mv = q = 2\frac{mM}{m+M} V\cos\theta \\ &\quad = 2\mu V\cos\theta \end{aligned}}$$

but nice insight

$$\cos\theta = \frac{1}{\sqrt{2}}\sqrt{1-\cos\theta^*} = \frac{1}{\sqrt{2}}\left(1 - \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right)^{1/2}$$

$$\cos\theta = \sin\frac{\theta^*}{2}$$

so $0 < \theta < \pi/2$

so $\boxed{\theta = \frac{\pi}{2} - \frac{\theta^*}{2}}$

1/3/4

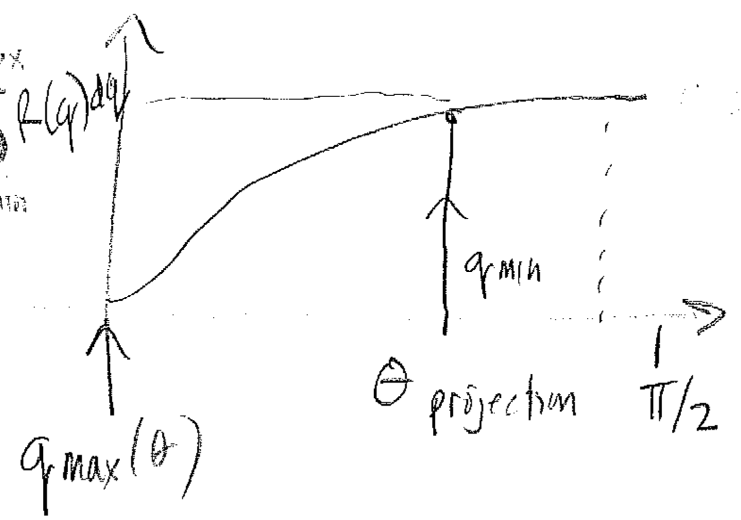
From $\theta = \frac{\pi}{2} - \frac{\theta^*}{2}$, you see:

$\theta = 0$ is $\theta^* = \pi$ (back scatter)

$\theta = \frac{\pi}{2}$ is $\theta^* = 0$ (no scatter)

What I did to reduce data:

$$I = \int_{q_{min}}^{q_{max}} R(q)\,dq$$



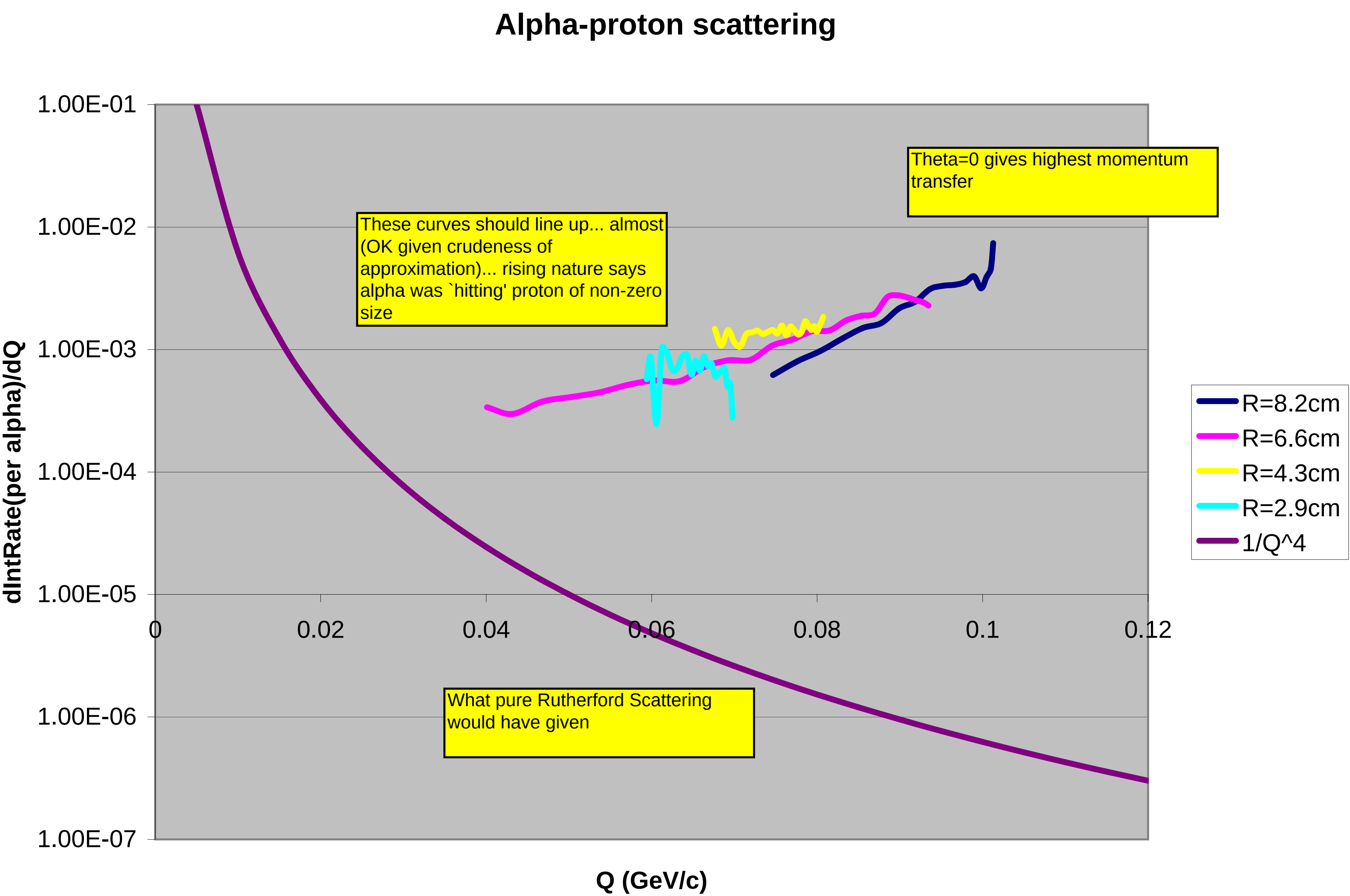
(neglect difference between $\theta_{projection}$ + θ)

$$R(q) = \frac{dI}{dq}$$

= what I plot

= "universal" curve, independent of q (almost true)

$\neq 1/q^4$; rising with q means high-q collisions more likely, "hitting" something, the nucleus!


Alpha-proton scattering
1.00E-01
1.00E-02
1.00E-03
1.00E-04
1.00E-05
1.00E-06
1.00E-07
dIntRate(per alpha)/dQ
0
0.02
0.04
0.06
0.08
0.1
0.12
Q (GeV/c)
These curves should line up... almost (OK given crudeness of approximation)... rising nature says alpha was `hitting' proton of non-zero size
Theta=0 gives highest momentum transfer
What pure Rutherford Scattering would have given
R=8.2cm
R=6.6cm
R=4.3cm
R=2.9cm
1/Q^4

$\boxed{2.}$ $\quad M_{if} \propto \underline{\tilde{\rho}(\vec{q})}\,\underline{\tilde{V}(\vec{q})}$

- "stuff" distribution.
- $\int \rho(\vec{x})\, d^3x = 1$

- interaction potential
- δ - function in physical space.
- constant in momentum space

$$\rho(\vec{x}) = N \quad \text{(normalization)}$$

when $\quad r < R(A) = 1.2A^{1/3}$

$$= 0 \qquad r \geq 1.2A^{1/3}$$

But $\quad \int \rho(\vec{x})\, d^3x = \frac{4\pi R^3}{3} N = 1$

$$\therefore\ N = \frac{3}{4\pi R^3}$$

$$\tilde{\rho}(\vec{q}) = \int d^3x\, e^{-i\vec{q}\cdot\vec{x}} \rho(\vec{x}) \qquad \hbar = 1$$

Note: $\tilde{\rho}(0) = \int d^3x\, \rho(\vec{x}) = 1$ (norm)

$$\tilde{\rho}(\vec{q}) = N \int_0^{2\pi}\int_{-1}^{1}\int_0^{R(A)} d\phi\, d\mu\, r^2 dr\; e^{-iqr\mu}$$

$$= 2\pi N \int_{-1}^{1}\int_{0}^{R(A)} d\mu\, r^2 dr\, e^{-iqr\mu}$$

$$= 2\pi N \int_{0}^{R(A)} r\cancel{r^2} dr \left(\frac{e^{-iqr\mu}}{-iq\cancel{r}}\right)\Bigg|_{-1}^{1}$$

$$= \underbrace{\frac{4\pi N}{q}\int_{0}^{R(A)} dr\, r \sin(qr)}_{\text{integrate by parts or look up}}$$

$$\tilde{\rho}(\vec{q}) = \frac{4\pi N}{q^3}\left[\sin(qR) - qR\cos(qR)\right]$$

check as $q \to 0$

$$\tilde{\rho}(0) \Rightarrow \frac{4\pi N}{q^3}\left[\cancel{qR} - \frac{1}{6}(qR)^3 - \cancel{qR} + \frac{1}{2}(qR)^3\right]$$

$$\tilde{\rho}(0) = \frac{4\pi}{3}R^3 \cdot N = \underline{1}$$

so another nice expression, plugging in for N

$$\boxed{\begin{aligned}\tilde{\rho}(\vec{q}) &= \frac{3}{(qR)^3}\left[\sin(qR) - (qR)\cos(qR)\right] \\ &= \frac{3}{qR}\, j_1(qR) \quad (\text{spherical Bessel function}\end{aligned}}$$

<u>note</u> to translate from the $\hbar = c = 1$ world..

$$qR \rightarrow \frac{cq\,R}{\hbar c}\,; \quad \text{put } cq \text{ in MeV}$$

<u>Silicon</u> : $A = 28$

$$R(A) = 1.2\,28^{1/3} = 3.64 \text{ fm}$$

$$\frac{R(A)}{\hbar c} = \frac{3.64}{197.3} = \frac{1}{54.1 \text{ MeV}}$$

$$\rho_{Si}(\vec{q}) = 3\left(\frac{54.1}{cq}\right)^3\left[\sin\left(\frac{cq}{54.1}\right) - \left(\frac{cq}{54.1}\right)\cos\left(\frac{cq}{54.1}\right)\right]$$

<u>Xenon</u> : $A = 132$

$$R(A) = 1.2(132)^{1/3} = 6.11 \text{ fm}$$

$$\frac{R(A)}{\hbar c} = \frac{6.11}{197.3} = \frac{1}{32.3 \text{ MeV}}$$

$$\rho_{Xe}(\vec{q}) = 3\left(\frac{32.3}{cq}\right)^3\left[\sin\left(\frac{cq}{32.3}\right) - \left(\frac{cq}{32.3}\right)\cos\left(\frac{cq}{32.3}\right)\right]$$

cq in MeV everywhere

Maximum Momentum Transfer:

$\simeq 2 p_0 \qquad p_0 \simeq m \cdot \beta c$

$\simeq 10^2 \cdot 10^3 \text{ MeV} \cdot \frac{300}{3 \cdot 10^5}$

$\simeq 100 \text{ MeV}$

$\simeq 200 \text{ MeV}$

Reality: should use reduced mass

Si: $\mu \cong \frac{(28 \cdot 0.93) 100}{100 + (28 \cdot 0.93)}$ 0.93 GeV/amu

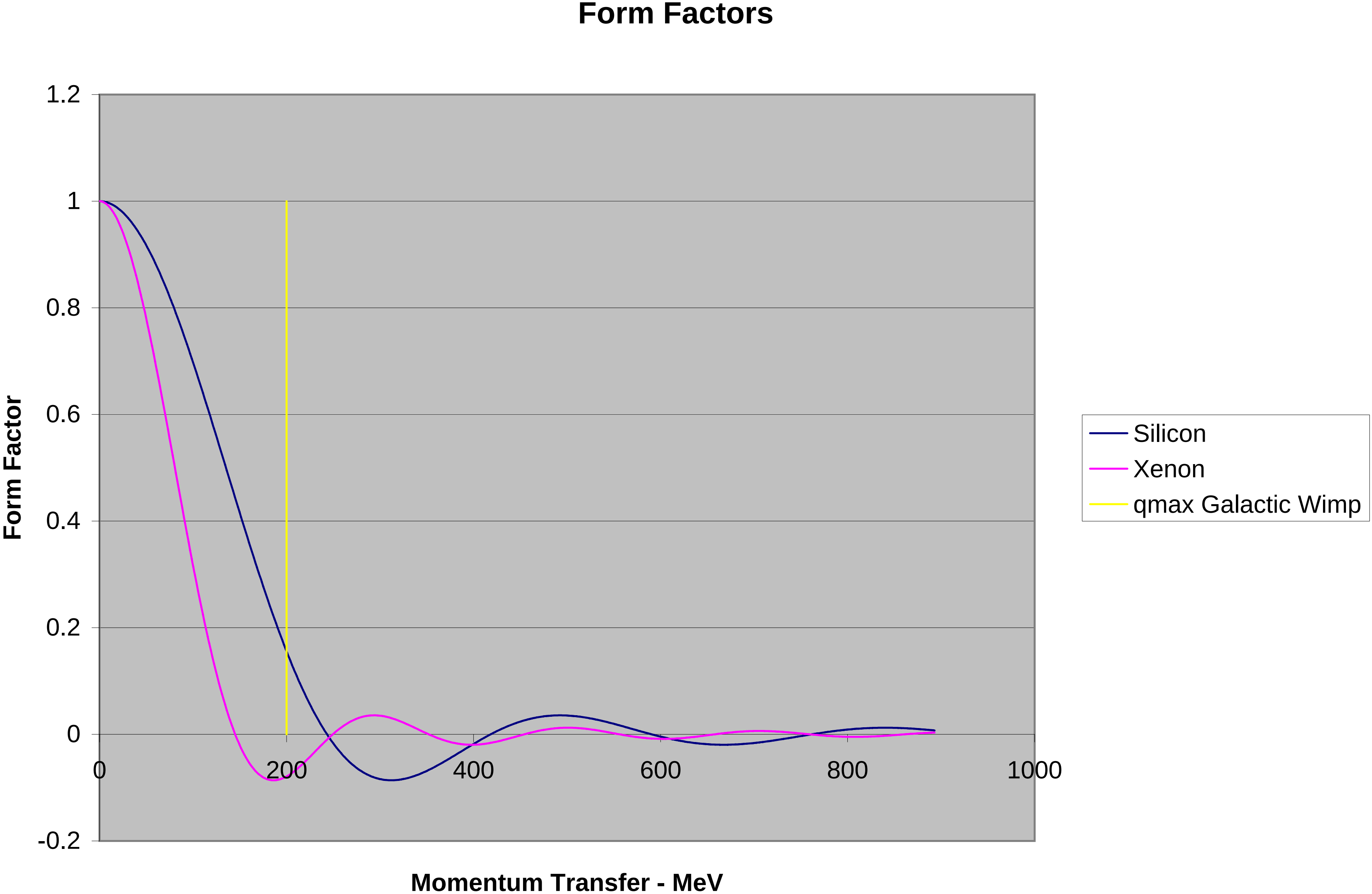
$\simeq 20 \text{ GeV}/c^2$

Xe $\mu \simeq \frac{(132 \cdot 0.93) \cdot 100}{100 + (132 \cdot 0.93)}$

$\simeq 55 \text{ GeV } c^2$

	q_{max}
Si	40 MeV
Xe	110 MeV

taking reduced mass into effect



Form Factors
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
Form Factor
0
200
400
600
800
1000
Momentum Transfer - MeV
Silicon
Xenon
qmax Galactic Wimp