CHAPTER 13

Collisions, Energy Loss, and Scattering of Charged Particles; Cherenkov and Transition Radiation

In this chapter we consider collisions between swiftly moving, charged particles, with special emphasis on the exchange of energy between collision partners and on the accompanying deflections from the incident direction. We also treat Cherenkov radiation and transition radiation, phenomena associated with charged particles in uniform motion through material media.

A fast charged particle incident on matter makes collisions with the atomic electrons and nuclei. If the particle is heavier than an electron (mu or pi meson, K meson, proton, etc.), the collisions with electrons and with nuclei have different consequences. The light electrons can take up appreciable amounts of energy from the incident particle without causing significant deflections, whereas the massive nuclei absorb very little energy but because of their greater charge cause scattering of the incident particle. Thus loss of energy by the incident particle occurs almost entirely in collisions with electrons. The deflection of the particle from its incident direction results, on the other hand, from essentially elastic collisions with the atomic nuclei. The scattering is confined to rather small angles, so that a heavy particle keeps a more or less straight-line path while losing energy until it nears the end of its range. For incident electrons both energy loss and scattering occur in collisions with the atomic electrons. Consequently the path is much less straight. After a short distance, electrons tend to diffuse into the material, rather than go in a rectilinear path.

The subject of energy loss and scattering is an important one and is discussed in several books (see references at the end of the chapter) where numerical tables and graphs are presented. Consequently our discussion emphasizes the physical ideas involved, rather than the exact numerical formulas. Indeed, a full quantum-mechanical treatment is needed to obtain exact results, even though all the essential features are classical or semiclassical in origin. All the orders of magnitude of the quantum effects are easily derivable from the uncertainty principle, as will be seen.

We begin by considering the simple problem of energy transfer to a free electron by a fast heavy particle. Then the effects of a binding force on the electron are explored, and the classical Bohr formula for energy loss is obtained. A description of quantum modifications and the effect of the polarization of the medium is followed by a discussion of the closely related phenomenon of Cherenkov radiation in transparent materials. Then the elastic scattering of incident particles by nuclei and multiple scattering are presented. Finally, we treat transition radiation by a particle passing from one medium to another of different optical properties.

13.1 *Energy Transfer in a Coulomb Collision Between Heavy Incident Particle and Stationary Free Electron; Energy Loss in Hard Collisions*

A swift particle of charge ze and mass M (energy $E = \gamma Mc^2$, momentum $P = \gamma\beta Mc$) collides with an atomic electron of charge $-e$ and mass m. For energetic collisions the binding of the electron in the atom can be neglected; the electron can be considered free and initially at rest in the laboratory. For all incident particles except electrons and positrons, $M \gg m$. Then the collision is best viewed as elastic Coulomb scattering in the rest frame of the incident particle. The well-known Rutherford scattering formula is

$$\frac{d\sigma}{d\Omega} = \left(\frac{ze^2}{2pv}\right)^2 \operatorname{cosec}^4 \frac{\theta}{2} \tag{13.1}$$

where $p = \gamma\beta mc$ and $v = \beta c$ are the momentum and speed of the electron in the rest frame of the heavy particle (exact in the limit $M/m \to \infty$). The cross section can be given a Lorentz-invariant form by relating the scattering angle to the 4-momentum transfer squared, $Q^2 = -(p - p')^2$. For elastic scattering, $Q^2 = 4p^2 \sin^2(\theta/2)$. The result is

$$\frac{d\sigma}{dQ^2} = 4\pi\left(\frac{ze^2}{\beta c Q^2}\right)^2 \tag{13.2}$$

where βc, the relative speed in each particle's rest frame, is found from $\beta^2 = 1 - (Mmc^2/P \cdot p)^2$.

The cross section for a given energy loss T by the incident particle, that is, the kinetic energy imparted to the initially stationary electron, is proportional to (13.2). If we evaluate the invariant Q^2 in the electron's rest frame, we find $Q^2 = 2mT$. With Q^2 replaced by $2mT$, (13.2) becomes

$$\frac{d\sigma}{dT} = \frac{2\pi z^2 e^4}{mc^2\beta^2 T^2} \tag{13.3}$$

Equation (13.3) is the cross section per unit energy interval for energy loss T by the massive incident particle in a Coulomb collision with a free stationary electron. Its range of validity for actual collisions in matter is

$$T_{\min} < T < T_{\max}$$

with $T_{\min}$ set by our neglect of binding ($T_{\min} \gtrsim \hbar\langle\omega\rangle$ where $\hbar\langle\omega\rangle$ is an estimate of the mean effective atomic binding energy) and $T_{\max}$ governed by kinematics. We can find $T_{\max}$ by recognizing that the most energetic collision in the rest frame of the incident particle occurs when the electron reverses its direction. After such a collision, the electron has energy $E' = \gamma mc^2$ and momentum $p' = \gamma\beta mc$ in the direction of the incident particle's velocity in the laboratory. The boost to the laboratory gives

$$T_{\max} = E - mc^2 = \gamma(E' + \beta cp') - mc^2 = 2\gamma^2\beta^2 mc^2 \tag{13.4}$$

We note in passing that (13.4) is not correct if the incident particle has too high an energy. The exact answer for $T_{\max}$ has a factor in the denominator, $D = 1 + 2mE/M^2c^2 + m^2/M^2$. For muons ($M/m \approx 207$), the denominator must be taken into account if the energy is comparable to 44 GeV or greater. For protons that energy is roughly 340 GeV. For equal masses, it is easy to see that $T_{\max} = (\gamma - 1)mc^2$.

When the spin of the electron is taken into account, there is a quantum-mechanical correction to the energy loss cross section, namely, a factor of $1 - \beta^2 \sin^2(\theta/2) = (1 - \beta^2\, T/T_{\max})$:

$$\left(\frac{d\sigma}{dT}\right)_{qm} = \frac{2\pi z^2 e^4}{mc^2\beta^2 T^2}\left(1 - \beta^2 \frac{T}{T_{\max}}\right) \tag{13.5}$$

The energy loss per unit distance in collisions with energy transfer greater than ε for a heavy particle passing through matter with N atoms per unit volume, each with Z electrons, is given by the integral,

$$\begin{aligned}\frac{dE}{dx}(T > \varepsilon) &= NZ \int_{\varepsilon}^{T_{\max}} T \frac{d\sigma}{dT}\, dT \\ &= 2\pi NZ \frac{z^2 e^4}{mc^2\beta^2}\left[\ln\left(\frac{2\gamma^2\beta^2 mc^2}{\varepsilon}\right) - \beta^2\right]\end{aligned} \tag{13.6}$$

In the result (13.6) we assumed $\varepsilon \ll T_{\max}$ and used (13.5) for the energy-transfer cross section. The small term, $-\beta^2$, in the square brackets is the relativistic spin contribution. Equation (13.6) represents the energy loss in close collisions. It is only valid provided $\varepsilon \gg \hbar\langle\omega\rangle$ because binding has been ignored.

An alternative, classical or semiclassical approach throws a different light on the physics of energy loss. In the rest frame of the heavy particle the incident electron approaches at impact parameter b. There is a one-to-one correspondence between b and the scattering angle θ (see Problem 13.1). The energy transfer T can be written as

$$T(b) = \frac{2z^2e^4}{mv^2} \cdot \frac{1}{b^2 + b_{\min}^{(c)2}}$$

with $b_{\min}^{(c)} = ze^2/pv$. For $b \gg b_{\min}^{(c)}$ the energy transfer varies as b^{-2}, implying that, if the energy transfer is greater than ε, the impact parameter must be less than the maximum,

$$b_{\max}^{(c)}(\varepsilon) \approx \left(\frac{2z^2e^4}{mv^2\varepsilon}\right)^{1/2}$$

When the heavy particle passes through matter it "sees" electrons at all possible impact parameters, with weighting according to the area of an annulus, $2\pi b\, db$. The classical energy loss per unit distance for collisions with transfer greater than ε is therefore

$$\frac{dE}{dx}(T > \varepsilon) = 2\pi NZ \int_0^{b_{\max}^{(c)}(\varepsilon)} T(b) b\, db = 2\pi NZ \frac{z^2e^4}{mc^2\beta^2} \ln\left[\left(\frac{b_{\max}^{(c)}(\varepsilon)}{b_{\min}^{(c)}}\right)^2\right] \tag{13.7}$$

Substitution of $b_{\max}$ and $b_{\min}$ leads directly to (13.6), apart from the relativistic spin correction. That we obtain the same result (for a spinless particle) quantum mechanically and classically is a consequence of the validity of the Rutherford cross section in both regimes.

If we wish to find a classical result for the *total* energy loss per unit distance, we must address the influence of atomic binding. Electronic binding can be characterized by the frequency of motion $\langle\omega\rangle$ or its reciprocal, the period. The incident heavy particle produces appreciable time-varying electromagnetic fields at the atom for a time $\Delta t \approx b/\gamma v$ [see (11.153)]. If the characteristic time Δt is long compared to the atomic period, the atom responds adiabatically—it stretches slowly during the encounter and returns to normal, without appreciable energy being transferred. On the other hand, if Δt is very short compared to the characteristic period, the electron can be treated as almost free. The dividing line is $\langle\omega\rangle\Delta t \approx 1$, implying a maximum effective impact parameter

$$b_{\max}^{(c)} \approx \frac{\gamma v}{\langle\omega\rangle} \tag{13.8}$$

beyond which no significant energy transfer is possible. Explicit illustration of this cutoff for a charge bound harmonically is found in Problems 13.2 and 13.3.

If (13.8) is used in (13.7) instead of $b_{\max}^{(c)}(\varepsilon)$, the total classical energy loss per unit distance is approximately

$$\left(\frac{dE}{dx}\right)_{\text{classical}} = 2\pi NZ \frac{z^2e^4}{mc^2\beta^2} \ln(B_c^2) \tag{13.9}$$

where

$$B_c = \lambda \frac{\gamma^2\beta^3 mc^3}{ze^2\langle\omega\rangle} = \lambda \frac{\gamma^2\beta^2 mc^2}{\eta\hbar\langle\omega\rangle} \tag{13.10}$$

In (13.10) we have inserted a numerical constant λ of the order of unity to allow for our uncertainty in $b_{\max}^{(c)}$. The parameter $\eta = ze^2/\hbar v$ is a characteristic of quantum-mechanical Coulomb scattering: $\eta \ll 1$ is the strongly quantum limit; $\eta \gg 1$ is the classical limit.

Equation (13.9) with (13.10) contains the essentials of the classical energy loss formula derived by Niels Bohr (1915). With many different electronic frequencies, $\langle\omega\rangle$ is the geometric mean of all the frequencies ω_j, weighted with the oscillator strength f_j:

$$Z \ln\langle\omega\rangle = \sum_j f_j \ln \omega_j \tag{13.11}$$

Equation (13.10) is valid for $\eta > 1$ (relatively slow alpha particles, heavy nuclei) but overestimates the energy loss when $\eta < 1$ (muons, protons, even fast alpha particles). We see below that when $\eta < 1$ the correct result sets $\eta = 1$ in (13.10).

13.2 Energy Loss from Soft Collisions; Total Energy Loss

The energy loss in collisions with energy transfers less than ε, including those small compared to electronic binding energies, really can be treated properly only by quantum mechanics, although after the fact we can "explain" the result in semiclassical language. The result, first obtained by Bethe (1930), is

$$\frac{dE}{dx}(T < \varepsilon) = 2\pi NZ \frac{z^2e^4}{mc^2\beta^2}\{\ln[B_q^2(\varepsilon)] - \beta^2\} \tag{13.12}$$

where

$$B_q(\varepsilon) = \frac{\gamma v (2m\varepsilon)^{1/2}}{\hbar\langle\omega\rangle} \tag{13.13}$$

The effective excitation energy $\hbar\langle\omega\rangle$ is given by (13.11), but now with the proper quantum-mechanical oscillator strengths and frequency differences for the atom, including the contribution from the continuum. The upper limit ε on the energy transfers is assumed to be beyond the limit of appreciable oscillator strength. Such a limit is consonant with the lower limit ε in Section 13.1, chosen to make the electron essentially free.

The total energy loss per unit length is given by the sum of (13.6) and (13.12):

$$\frac{dE}{dx} = 4\pi NZ \frac{z^2 e^4}{mc^2\beta^2}\{\ln(B_q) - \beta^2\} \tag{13.14}$$

where

$$B_q = \frac{2\gamma^2\beta^2 mc^2}{\hbar\langle\omega\rangle} \tag{13.15}$$

The general behavior of both the classical and quantum-mechanical energy loss formulas is illustrated in Fig. 13.1. They are functions only of the speed of the incident heavy particle, the mass and charge of the electron, and the mean excitation energy $\hbar\langle\omega\rangle$. For low energies ($\gamma\beta < 1$) the main dependence is as $1/\beta^2$, while at high energies the slow variation is proportional to $\ln(\gamma)$. The minimum value of dE/dx occurs at $\gamma\beta \approx 3$. The coefficient in (13.12) and (13.14) is numerically equal to 0.150 $z^2(2Z/A)\rho$ MeV/cm, where Z is the atomic number and A the mass number of the material, while ρ (g/cm^3) is its density. Since $2Z/A \approx 1$, the energy loss in MeV$\cdot$(cm^2/g) for a singly charged particle in aluminum is approximately what is shown in Fig. 13.1. For aluminum the minimum energy loss is roughly 1.7 MeV$\cdot$(cm^2/g); for lead, it is 1.2 MeV$\cdot$(cm^2/g). At high energies corrections to the behavior in Fig. 13.1 occur. The energy loss becomes heavy-particle specific, through the mass-dependent denominator D in $T_{\max}$, and

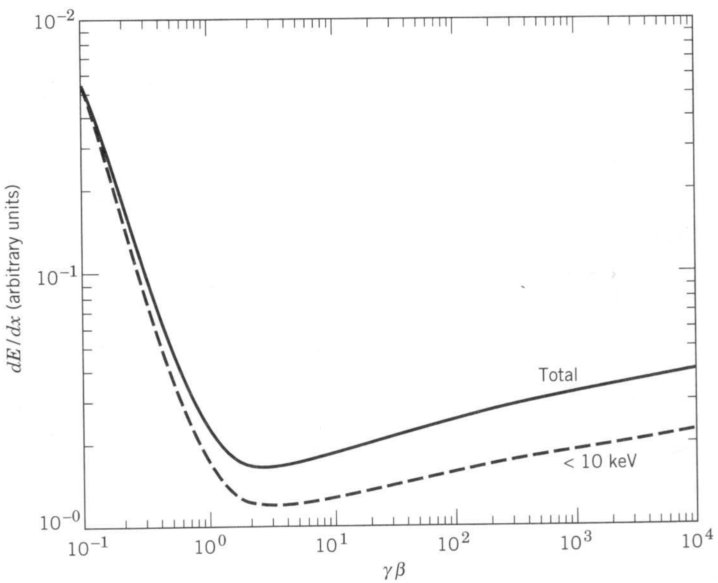

Figure 13.1 Energy loss as a function of $\gamma\beta$ of the incident heavy particle. The solid curve is the total energy loss (13.14) with $\hbar\langle\omega\rangle = 160$ eV (aluminum). The dashed curve is the energy loss in soft collisions (13.12) with $\varepsilon = 10$ keV. The ordinate scale corresponds to the curly-bracketed quantities in (13.12) and (13.14), multiplied by 0.15.

has a different energy variation and dependence on the material, because of the density effect discussed in Section 13.3.

The restricted energy loss shown in Fig. 13.1 is applicable to the energy loss inferred from tracks in photographic emulsions. Electrons with energies greater than about 10 keV have sufficient range to escape from silver bromide grains. The density of blackening along a track is therefore related to the restricted energy loss. Note that it increases more slowly than the total for large $\gamma\beta$—as $\ln(\gamma)$ rather than $\ln(\gamma^2)$. A semiclassical explanation is given below.

Comparison of B_q with the classical B_c (13.10) shows that their ratio is $\eta = ze^2/\hbar v$. To understand how this factor arises, we turn to semiclassical arguments. B_c is the ratio of $b_{\max}^{(c)}$ (13.8) to $b_{\min}^{(c)} = ze^2/\gamma m v^2$. The uncertainty principle dictates a different $b_{\min}$ for $\eta < 1$. In the rest frame of the heavy particle the electron has momentum $p = \gamma m v$. If it is described by a transversely localized wave packet (to define its impact parameter as well as possible), the spread in transverse momenta Δp around zero must satisfy $\Delta p \ll p$; otherwise, its longitudinal direction would be ill-defined. This limit on Δp translates into an uncertainty Δb in impact parameter, $\Delta b \gg \hbar/p$, or in other words, an effective quantum-mechanical lower limit,

$$b_{\min}^{(q)} = \frac{\hbar}{\gamma m v} \tag{13.16}$$

Evidently, in calculating energy loss as an integral over impact parameters, the larger of the two minimum impact parameters should be used. The ratio $b_{\min}^{(c)}/b_{\min}^{(q)} = \eta$. When $\eta > 1$, the classical lower limit applies; for $\eta < 1$, (13.16) applies and (13.15) is the correct expression for B.

The value of $B_q(\varepsilon)$ in (13.12) can also be understood in terms of impact parameters. The soft collisions contributing to (13.12) come semiclassically from the more distant collisions. The momentum transfer δp to the struck electron in such collisions is related to the energy transfer T according to $\delta p = (2mT)^{1/2}$. On the other hand, the localized electron wave packet has a spread Δp in transverse momenta. To be certain that the collision produces an energy transfer less than ε, we must have $\Delta p < \delta p_{\max} = (2m\varepsilon)^{1/2}$, hence $\Delta b > \hbar/(2m\varepsilon)^{1/2}$. The effective minimum impact parameter for soft collisions with energy transfer less than ε is therefore

$$b_{\min}^{(q)}(\varepsilon) \approx \frac{\hbar}{(2m\varepsilon)^{1/2}} \tag{13.17}$$

For collisions so limited in impact parameter between (13.17) and $b_{\max} = \gamma v/\langle\omega\rangle$, we find

$$B_q(\varepsilon) \approx \frac{\gamma v(2m\varepsilon)^{1/2}}{\hbar\langle\omega\rangle}$$

in agreement with Bethe's result.

The semiclassical discussion of the minimum and maximum impact parameters elucidates the reason for the difference in the logarithmic growth between the restricted and total energy losses. At high energies the dominant energy dependence is through $dE/dx \propto \ln(B) \approx \ln(b_{\max}/b_{\min})$. For the total energy loss, the maximum impact parameter is proportional to γ, while the quantum-mechanical minimum impact parameter (13.16) is inversely proportional to γ. The ratio varies as γ^2. For energy loss restricted to energy transfers less than ε, the minimum impact parameter (13.17) is independent of γ, leading to $B_q(\varepsilon) \propto \gamma$.

Despite its attractiveness in making clear the physics, the semiclassical description in terms of impact parameters contains a conceptual difficulty that warrants discussion. Classically, the energy transfer T in each collision is related directly to the impact parameter b. When $b \gg b_{\text{min}}^{(c)}$, $T(b) \approx 2z^2e^4/mv^2b^2$ (Problem 13.1). With increasing b the energy transfer decreases rapidly until at $b = b_{\text{max}} \approx \gamma v/\langle\omega\rangle$ it becomes

$$T(b_{\text{max}}) \approx \frac{z^2}{\gamma^2}\left(\frac{v_0}{v}\right)^4\left(\frac{\hbar\langle\omega\rangle}{I_{\text{H}}}\right)\hbar\langle\omega\rangle \tag{13.18}$$

Here $v_0 = c/137$ is the orbital speed of an electron in the ground state of hydrogen and $I_{\text{H}} = 13.6$ eV its ionization potential. Since empirically $\hbar\langle\omega\rangle \le ZI_{\text{H}}$, we see that for a fast particle ($v \gg v_0$) the classical energy transfer (13.18) is much smaller than the ionization potential, indeed, smaller than the minimum possible atomic excitation.

We know, however, that energy must be transferred to the atom in discrete quantum jumps. A tiny amount of energy such as (13.18) simply cannot be absorbed by the atom. We might argue that the classical expression for $T(b)$ should be employed only if it is large compared to some typical excitation energy of the atom. This requirement would set quite a different upper limit on the impact parameters from $b_{\text{max}} \approx \gamma v/\langle\omega\rangle$ and lead to wrong results. Could b_{max} nevertheless be wrong? After all, it came from consideration of the time dependence of the electric and magnetic fields (11.152), without consideration of the system being affected. No, time-dependent perturbations of a quantum system cause significant excitation only if they possess appreciable Fourier components with frequencies comparable to $1/\hbar$ times the lowest energy difference. That was the "adiabatic" argument that led to b_{max} in the first place. The solution to this conundrum lies in another direction. The classical expressions must be interpreted in a statistical sense.

The classical concept of the transfer of a small amount of energy in every collision is incorrect quantum-mechanically. Instead, while *on the average* over many collisions, a small energy is transferred, the small average results from appreciable amounts of energy transferred in a very small fraction of those collisions. In most collisions no energy is transferred. It is only in a statistical sense that the quantum-mechanical mechanism of discrete energy transfers and the classical process with a continuum of possible energy transfers can be reconciled. The detailed numerical agreement for the averages (but not for the individual amounts) stems from the quantum-mechanical definitions of the oscillator strengths f_j and resonant frequencies ω_j entering $\langle\omega\rangle$. A meaningful semiclassical description requires (a) the statistical interpretation and (b) the use of the uncertainty principle to set appropriate minimum impact parameters.

The discussion so far has been about energy loss by a heavy particle of mass $M \gg m$. For electrons ($M = m$), kinematic modifications occur in the energy loss in hard collisions. The maximum energy loss is $T_{\text{max}} = (\gamma - 1)mc^2$. The argument of the logarithm in (13.6) becomes $(\gamma - 1)mc^2/\varepsilon$. The Bethe expression (13.12) for soft collisions remains the same. The total energy loss for electrons therefore has B_q (13.15) replaced by

$$B_q(\text{electrons}) = \frac{\sqrt{2}\,\gamma\beta\sqrt{\gamma - 1}\,mc^2}{\hbar\langle\omega\rangle} \approx \frac{\sqrt{2}\,\gamma^{3/2}\,mc^2}{\hbar\langle\omega\rangle} \tag{13.19}$$

the last form applicable for relativistic energies. There are spin and exchange effects in addition to the kinematic change, but the dominant effect is in the argument of the logarithm; the other effects only contribute to the added constant.

The expressions for dE/dx represent the *average* collisional energy loss per unit distance by a particle traversing matter. Because the number of collisions per unit distance is finite, even though large, and the spectrum of possible energy transfers in individual collisions is wide, there are fluctuations around the average. These fluctuations produce straggling in energy or range for a particle traversing a certain thickness of matter. If the number of collisions is large enough and the mean energy loss not too great, the final energies of a beam of initially monoenergetic particles of energy E_0 are distributed in Gaussian fashion about the mean $\overline{E}$. With Poisson statistics for the number of collisions producing a given energy transfer T, it can be shown (see, e.g., *Bohr*, Section 2.3, or *Rossi*, Section 2.7) that the mean square deviation in energy from the mean is

$$\Omega^2 = 2\pi NZz^2e^4(\gamma^2 + 1)t \tag{13.20}$$

where t is the thickness traversed. This result holds provided $\Omega \ll \overline{E}$ *and* $\Omega \ll (E_0 - \overline{E})$, and also $\Omega \gg T_{\text{max}} \approx 2\gamma^2\beta^2mc^2$. For ultrarelativistic particles the last condition ultimately fails. Then the distribution in energies is not Gaussian, but is described by the Landau curve. The interested reader may consult the references at the end of the chapter for further details.

13.3 Density Effect in Collisional Energy Loss

For particles that are not too relativistic, the observed energy loss is given accurately by (13.14) [or by (13.9) if $\eta > 1$] for particles of all kinds in media of all types. For ultrarelativistic particles, however, the observed energy loss is less than predicted by (13.14), especially for dense substances. In terms of Fig. 13.1 of (dE/dx), the observed energy loss increases beyond the minimum with a slope of roughly one-half that of the theoretical curve, corresponding to only one power of γ in the argument of the logarithm in (13.14) instead of two. In photographic emulsions the energy loss, as measured from grain densities, barely increases above the minimum to a plateau extending to the highest known energies. This again corresponds to a reduction of one power of γ, this time in $B_q(\epsilon)$ (13.13).

This reduction in energy loss, known as the density effect, was first treated theoretically by Fermi (1940). In our discussion so far we have tacitly made one assumption that is not valid in dense substances. We have assumed that it is legitimate to calculate the effect of the incident particle's fields on one electron in one atom at a time, and then sum up incoherently the energy transfers to all the electrons in all the atoms with $b_{\text{min}} < b < b_{\text{max}}$. Now b_{max} is very large compared to atomic dimensions, especially for large γ. Consequently in dense media there are many atoms lying between the incident particle's trajectory and the typical atom in question if b is comparable to b_{max}. These atoms, influenced themselves by the fast particle's fields, will produce perturbing fields at the chosen atom's position, modifying its response to the fields of the fast particle. Said in another way, in dense media the dielectric polarization of the material alters the particle's fields from their free-space values to those characteristic of macroscopic fields in a dielectric. This modification of the fields due to polarization of the medium must be taken into account in calculating the energy transferred in distant collisions. For close collisions the incident particle interacts with only one atom at a time. Then the free-particle calculation without polarization effects will apply. The dividing impact parameter between close and distant collisions is of the order of atomic dimensions. Since the joining of two logarithms is involved in calculating the sum, the dividing value of b need not be specified with great precision.

We will determine the energy loss in distant collisions ($b \geq a$), assuming that the fields in the medium can be calculated in the continuum approximation of a macroscopic dielectric constant $\epsilon(\omega)$. If a is of the order of atomic dimensions, this approximation will not be good for the closest of the distant collisions, but will be valid for the great bulk of the collisions.

The problem of finding the electric field in the medium due to the incident fast particle moving with constant velocity can be solved most readily by Fourier transforms. If the potentials $A_\mu(x)$ and source density $J_\mu(x)$ are transformed in space and time according to the general rule,

$$F(\mathbf{x}, t) = \frac{1}{(2\pi)^2} \int d^3k \int d\omega\, F(\mathbf{k}, \omega) e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} \tag{13.21}$$

then the transformed wave equations become

$$\begin{aligned} \left[k^2 - \frac{\omega^2}{c^2}\epsilon(\omega)\right]\Phi(\mathbf{k}, \omega) &= \frac{4\pi}{\epsilon(\omega)}\rho(\mathbf{k}, \omega) \\ \left[k^2 - \frac{\omega^2}{c^2}\epsilon(\omega)\right]\mathbf{A}(\mathbf{k}, \omega) &= \frac{4\pi}{c}\mathbf{J}(\mathbf{k}, \omega) \end{aligned} \tag{13.22}$$

The dielectric constant $\epsilon(\omega)$ appears characteristically in positions dictated by the presence of $\mathbf{D}$ in the Maxwell equations. The Fourier transforms of

$$\begin{aligned} \rho(\mathbf{x}, t) &= ze\,\delta(\mathbf{x} - \mathbf{v}t) \\ \text{and} \qquad \mathbf{J}(\mathbf{x}, t) &= \mathbf{v}\rho(\mathbf{x}, t) \end{aligned} \tag{13.23}$$

are readily found to be

$$\begin{aligned} \rho(\mathbf{k}, \omega) &= \frac{ze}{2\pi}\delta(\omega - \mathbf{k}\cdot\mathbf{v}) \\ \mathbf{J}(\mathbf{k}, \omega) &= \mathbf{v}\rho(\mathbf{k}, \omega) \end{aligned} \tag{13.24}$$

From (13.22) we see that the Fourier transforms of the potentials are

$$\begin{aligned} \Phi(\mathbf{k}, \omega) &= \frac{2ze}{\epsilon(\omega)} \cdot \frac{\delta(\omega - \mathbf{k}\cdot\mathbf{v})}{k^2 - \dfrac{\omega^2}{c^2}\epsilon(\omega)} \\ \text{and} \qquad \mathbf{A}(\mathbf{k}, \omega) &= \epsilon(\omega)\frac{\mathbf{v}}{c}\Phi(\mathbf{k}, \omega) \end{aligned} \tag{13.25}$$

From the definitions of the electromagnetic fields in terms of the potentials we obtain their Fourier transforms:

$$\begin{aligned} \mathbf{E}(\mathbf{k}, \omega) &= i\left[\frac{\omega\epsilon(\omega)}{c}\frac{\mathbf{v}}{c} - \mathbf{k}\right]\Phi(\mathbf{k}, \omega) \\ \mathbf{B}(\mathbf{k}, \omega) &= i\epsilon(\omega)\mathbf{k} \times \frac{\mathbf{v}}{c}\Phi(\mathbf{k}, \omega) \end{aligned} \tag{13.26}$$

In calculating the energy loss to an electron in an atom at impact parameter b, we evaluate

$$\Delta E = -e\int_{-\infty}^{\infty} \mathbf{v}\cdot\mathbf{E}\, dt = 2e\,\mathrm{Re}\int_0^{\infty} i\omega\mathbf{x}(\omega)\cdot\mathbf{E}^*(\omega)\, d\omega \tag{13.27}$$

where $\mathbf{x}(\omega)$ is the Fourier transform in time of the electron's coordinate and $\mathbf{E}(\omega)$ is the Fourier transform in time of the electromagnetic fields at a perpendicular distance b from the path of the particle moving along the x axis. Thus the required electric field is

$$\mathbf{E}(\omega) = \frac{1}{(2\pi)^{3/2}}\int d^3k\, \mathbf{E}(\mathbf{k}, \omega) e^{ibk_2} \tag{13.28}$$

where the observation point has coordinates $(0, b, 0)$. To illustrate the determination of $\mathbf{E}(\omega)$ we consider the calculation of $E_1(\omega)$, the component of $\mathbf{E}$ parallel to $\mathbf{v}$. Inserting the explicit forms from (13.25) and (13.26), we obtain

$$E_1(\omega) = \frac{2ize}{\epsilon(\omega)(2\pi)^{3/2}}\int d^3k e^{ibk_2}\left[\frac{\omega\epsilon(\omega)v}{c^2} - k_1\right]\frac{\delta(\omega - vk_1)}{k^2 - \dfrac{\omega^2}{c^2}\epsilon(\omega)} \tag{13.29}$$

The integral over dk_1 can be done immediately. Then

$$E_1(\omega) = -\frac{2ize\omega}{(2\pi)^{3/2}v^2}\left[\frac{1}{\epsilon(\omega)} - \beta^2\right]\int_{-\infty}^{\infty} dk_2\, e^{ibk_2}\int_{-\infty}^{\infty}\frac{dk_3}{k_2^2 + k_3^2 + \lambda^2}$$

where

$$\lambda^2 = \frac{\omega^2}{v^2} - \frac{\omega^2}{c^2}\epsilon(\omega) = \frac{\omega^2}{v^2}[1 - \beta^2\epsilon(\omega)] \tag{13.30}$$

The integral over dk_3 has the value $\pi/(\lambda^2 + k_2^2)^{1/2}$, so that $E_1(\omega)$ can be written

$$E_1(\omega) = -\frac{ize\omega}{\sqrt{2\pi}\,v^2}\left[\frac{1}{\epsilon(\omega)} - \beta^2\right]\int_{-\infty}^{\infty}\frac{e^{ibk_2}}{(\lambda^2 + k_2^2)^{1/2}}\, dk_2 \tag{13.31}$$

The remaining integral is a representation of a modified Bessel function.* The result is

$$E_1(\omega) = -\frac{ize\omega}{v^2}\left(\frac{2}{\pi}\right)^{1/2}\left[\frac{1}{\epsilon(\omega)} - \beta^2\right]K_0(\lambda b) \tag{13.32}$$

*See, for example, *Abramowitz and Stegun* (p. 376, formula 9.6.25); *Magnus, Oberhettinger, and Soni* (Chapter XI), or Bateman Manuscript Project, *Table of Integral Transforms*, Vol. 1 (Chapters I–III).

where the square root of (13.30) is chosen so that λ lies in the fourth quadrant. A similar calculation yields the other fields:

$$\left.\begin{aligned} E_2(\omega) &= \frac{ze}{v}\left(\frac{2}{\pi}\right)^{1/2}\frac{\lambda}{\epsilon(\omega)}K_1(\lambda b) \\ B_3(\omega) &= \epsilon(\omega)\beta E_2(\omega) \end{aligned}\right\} \tag{13.33}$$

In the limit $\epsilon(\omega) \to 1$ it is easily seen that fields (13.32) and (13.33) reduce to the results of Problem 13.3.

To find the energy transferred to the atom at impact parameter b we merely write down the generalization of (13.27):

$$\Delta E(b) = 2e \sum_j f_j \,\text{Re} \int_0^\infty i\omega \mathbf{x}_j(\omega) \cdot \mathbf{E}^*(\omega)\, d\omega$$

where $\mathbf{x}_j(\omega)$ is the amplitude of the jth type of electron in the atom. Rather than use (7.50) for $\mathbf{x}_j(\omega)$ we express the sum of dipole moments in terms of the molecular polarizability and so the dielectric constant. Thus

$$-e \sum_j f_j \mathbf{x}_j(\omega) = \frac{1}{4\pi N}[\epsilon(\omega) - 1]\mathbf{E}(\omega)$$

where N is the number of atoms per unit volume. Then the energy transfer can be written

$$\Delta E(b) = \frac{1}{2\pi N}\,\text{Re} \int_0^\infty -i\omega\epsilon(\omega)\,|\mathbf{E}(\omega)|^2\, d\omega \tag{13.34}$$

The energy loss per unit distance in collisions with impact parameter $b \geq a$ is evidently

$$\left(\frac{dE}{dx}\right)_{b>a} = 2\pi N \int_a^\infty \Delta E(b) b\, db \tag{13.35}$$

If fields (13.32) and (13.33) are inserted into (13.34) and (13.35), we find, after some calculation, the expression due to Fermi,

$$\left(\frac{dE}{dx}\right)_{b>a} = \frac{2}{\pi}\frac{(ze)^2}{v^2}\,\text{Re} \int_0^\infty i\omega\lambda^* a K_1(\lambda^* a) K_0(\lambda a)\left(\frac{1}{\epsilon(\omega)} - \beta^2\right) d\omega \tag{13.36}$$

where λ is given by (13.30). This result can be obtained more elegantly by calculating the electromagnetic energy flow through a cylinder of radius a around the path of the incident particle. By conservation of energy this is the energy lost per unit time by the incident particle. Thus

$$\left(\frac{dE}{dx}\right)_{b>a} = \frac{1}{v}\frac{dE}{dt} = -\frac{c}{4\pi v}\int_{-\infty}^\infty 2\pi a B_3 E_1\, dx$$

The integral over dx at one instant of time is equivalent to an integral at one point on the cylinder over all time. Using $dx = v\, dt$, we have

$$\left(\frac{dE}{dx}\right)_{b>a} = -\frac{ca}{2}\int_{-\infty}^\infty B_3(t)E_1(t)\, dt$$

In the standard way this can be converted into a frequency integral,

$$\left(\frac{dE}{dx}\right)_{b>a} = -ca\,\text{Re}\int_0^\infty B_3^*(\omega)E_1(\omega)\, d\omega \tag{13.37}$$

With fields (13.32) and (13.33) this gives the Fermi result (13.36).

The Fermi expression (13.36) bears little resemblance to our earlier results for energy loss. But under conditions where polarization effects are unimportant it yields the same results as before. For example, for nonrelativistic particles ($\beta \ll 1$) it is clear from (13.30) that $\lambda \simeq \omega/v$, independent of $\epsilon(\omega)$. Then in (13.36) the modified Bessel functions are real. Only the imaginary part of $1/\epsilon(\omega)$ contributes to the integral. If we neglect the polarization correction of Section 4.5 to the internal field at an atom, the dielectric constant can be written

$$\epsilon(\omega) \simeq 1 + \frac{4\pi N e^2}{m}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\Gamma_j} \tag{13.38}$$

where we have used the dipole moment expression (7.50). Assuming that the second term is small, the imaginary part of $1/\epsilon(\omega)$ can be readily calculated and substituted into (13.36). Then the integral over $d\omega$ can be performed in the narrow-resonance approximation. If the small-argument limits of the Bessel functions are used, the nonrelativistic form of (13.9) emerges, with $B_c = v/a\langle\omega\rangle$. If the departure of λ from $\omega/\gamma v$ in (13.30) is neglected, (13.9) emerges with $B_c = \gamma v/a\langle\omega\rangle$.

The density effect evidently comes from the presence of complex arguments in the modified Bessel functions, corresponding to taking into account $\epsilon(\omega)$ in (13.30). Since $\epsilon(\omega)$ there is multiplied by β^2, it is clear that the density effect can be really important only at high energies. The detailed calculations for all energies with some explicit expression such as (13.38) for $\epsilon(\omega)$ are quite complicated and not particularly informative. We content ourselves with the extreme relativistic limit ($\beta \simeq 1$). Furthermore, since the important frequencies in the integral over $d\omega$ are optical frequencies and the radius a is of the order of atomic dimensions, $|\lambda a| \sim (\omega a/c) \ll 1$. Consequently we can approximate the Bessel functions by their small-argument limits (3.103). Then in the relativistic limit the Fermi expression (13.36) is

$$\left(\frac{dE}{dx}\right)_{b>a} \simeq \frac{2}{\pi}\frac{(ze)^2}{c^2}\,\text{Re}\int_0^\infty i\omega\left(\frac{1}{\epsilon(\omega)} - 1\right)\left\{\ln\left(\frac{1.123c}{\omega a}\right) - \frac{1}{2}\ln[1 - \epsilon(\omega)]\right\} d\omega \tag{13.39}$$

It is worthwhile right here to point out that the argument of the second logarithm is actually $[1 - \beta^2\epsilon(\omega)]$. In the limit $\epsilon = 1$, this log term gives a factor γ in the combined logarithm, corresponding to the old result (13.9). Provided $\epsilon(\omega) \neq 1$, we can write this factor as $[1 - \epsilon(\omega)]$, thereby removing one power of γ from the logarithm, in agreement with experiment.

The integral in (13.39) with $\epsilon(\omega)$ given by (13.38) can be performed most easily by using Cauchy's theorem to change the integral over positive real ω to one over positive imaginary ω, minus one over a quarter-circle at infinity. The integral along the imaginary axis gives no contribution. Provided the Γ_j in (13.38) are assumed constant, the result of the integration over the quarter-circle can be written in the simple form:

$$\left(\frac{dE}{dx}\right)_{b>a} = \frac{(ze)^2\omega_p^2}{c^2}\ln\left(\frac{1.123c}{a\omega_p}\right) \tag{13.40}$$

where ω_p is the electronic plasma frequency

$$\omega_p^2 = \frac{4\pi NZe^2}{m} \tag{13.41}$$

The corresponding relativistic expression without the density effect is

$$\left(\frac{dE}{dx}\right)_{b>a} \simeq \frac{(ze)^2\omega_p^2}{c^2}\ln\left(\frac{1.123\gamma c}{a\langle\omega\rangle}\right) \tag{13.42}$$

We see that the density effect produces a simplification in that the asymptotic energy loss no longer depends on the details of atomic structure through $\langle\omega\rangle$ (13.11), but only on the number of electrons per unit volume through ω_p. Two substances having very different atomic structures will produce the same energy loss for ultrarelativistic particles provided their densities are such that the density of electrons is the same in each.

Since there are numerous calculated curves of energy loss based on Bethe's formula (13.14), it is often convenient to tabulate the decrease in energy loss due to the density effect. This is just the difference between (13.40) and (13.42):

$$\lim_{\beta\to 1}\Delta\left(\frac{dE}{dx}\right) = -\frac{(ze)^2\omega_p^2}{c^2}\ln\left(\frac{\gamma\omega_p}{\langle\omega\rangle}\right) \tag{13.43}$$

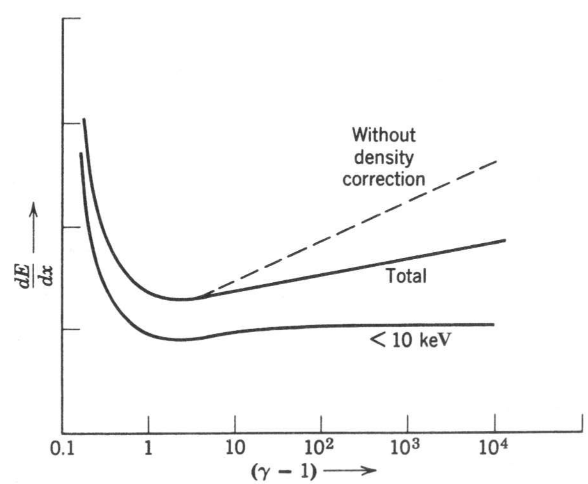

Figure 13.2 Energy loss, including the density effect. The dashed curve is the total energy loss without density correction. The solid curves have the density effect incorporated, the upper one being the total energy loss and the lower one the energy loss due to individual energy transfers of less than 10 keV.

For photographic emulsions, the relevant energy loss is given by (13.12) and (13.13) with $\varepsilon \simeq 10$ keV. With the density correction applied, this becomes constant at high energies with the value,

$$\frac{dE(\epsilon)}{dx} \to \frac{(ze)^2\omega_p^2}{2c^2}\ln\left(\frac{2mc^2\varepsilon}{\hbar^2\omega_p^2}\right) \tag{13.44}$$

For silver bromide, $\hbar\omega_p \simeq 48$ eV. Then for singly charged particles (13.44), divided by the density, has the value of approximately 1.02 MeV $\cdot$ (cm^2/g). This energy loss is in good agreement with experiment, and corresponds to an increase above the minimum value of less than 10%. Figure 13.2 shows total energy loss and loss from transfers of less than 10 keV for a typical substance. The dashed curve is the Bethe curve for total energy loss without correction for density effect.

13.4 *Cherenkov Radiation*

The density effect in energy loss is intimately connected to the coherent response of a medium to the passage of a relativistic particle that causes the emission of Cherenkov radiation. They are, in fact, the same phenomenon in different limiting circumstances. The expression (13.36), or better, (13.37), represents the energy lost by the particle into regions a distance greater than $b = a$ away from its path. By varying a we can examine how the energy is deposited throughout the medium. In (13.39) we have considered a to be atomic dimensions and assumed $|\lambda a| \ll 1$. Now we take the opposite limit. If $|\lambda a| \gg 1$, the modified Bessel functions can be approximated by their asymptotic forms. Then the fields (13.32) and (13.33) become

$$\begin{aligned} E_1(\omega, b) &\to i\frac{ze\omega}{c^2}\left[1 - \frac{1}{\beta^2\epsilon(\omega)}\right]\frac{e^{-\lambda b}}{\sqrt{\lambda b}} \\ E_2(\omega, b) &\to \frac{ze}{v\epsilon(\omega)}\sqrt{\frac{\lambda}{b}}\,e^{-\lambda b} \\ B_3(\omega, b) &\to \beta\epsilon(\omega)E_2(\omega, b) \end{aligned} \tag{13.45}$$

The integrand in (13.37) in this limit is

$$(-caB_3^*E_1) \to \frac{z^2e^2}{c^2}\left(-i\sqrt{\frac{\lambda^*}{\lambda}}\right)\omega\left[1 - \frac{1}{\beta^2\epsilon(\omega)}\right]e^{-(\lambda+\lambda^*)a} \tag{13.46}$$

The real part of this expression, integrated over frequencies, gives the energy deposited far from the path of the particle. If λ has a positive real part, as is generally true, the exponential factor in (13.46) will cause the expression to vanish rapidly at large distances. All the energy is deposited near the path. This is not true only when λ *is purely imaginary*. Then the exponential is unity; the expression is independent of a; *some of the energy escapes* to infinity *as radiation*. From (13.30) it can be seen that λ can be purely imaginary if $\epsilon(\omega)$ is real (no absorption) and $\beta^2\epsilon(\omega) > 1$. Actually, mild absorption can be allowed for, but