Work Energy Theorem in One D

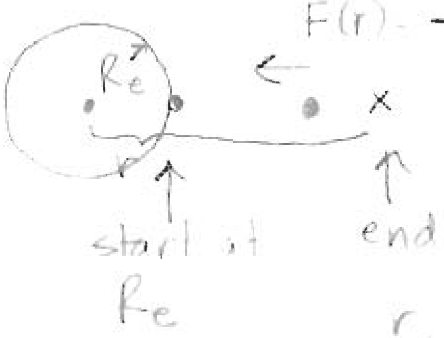
$\frac{1}{2}mv^2 = \text{Kinetic Energy} = K$

<u>units</u>: $kg \cdot \frac{m^2}{s^2} \equiv \text{Joule}$

$$\frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2 = K_b - K_a = \underbrace{\int_{x_a}^{x_b} F(x)\,dx}_{\text{work } W_{ba}}$$

$K_b - K_a = W_{ba}$ "Work Energy Theorem"

$F(r) = -\frac{GM_e m}{r^2}$ ($-$ means directed toward origin)



$$K(r) - K(R_e) = -GM_e m \int_{R_e}^{r} \frac{dr}{r^2}$$

r, highest point

$$= -GM_e m \left(-\frac{1}{r}\right)\Big|_{R_e}^{r}$$

$$\frac{1}{2}m\left(v^2(r) - v_0^2\right) = GM_e m\left(\frac{1}{r} - \frac{1}{R_e}\right)$$

$\rightarrow v(r) = 0$ when at highest point $r = r_{max}$

$$v_0^2 = 2M_e G\left(\frac{1}{R_e} - \frac{1}{r_{max}}\right)$$

$$= 2\underbrace{\left(\frac{M_e G}{R_e^2}\right)}_{g} R_e\left(1 - \frac{R_e}{r_{max}}\right)$$

$$\boxed{v_0^2 = 2gR_e\left(1 - \frac{R_e}{r_{max}}\right)}$$

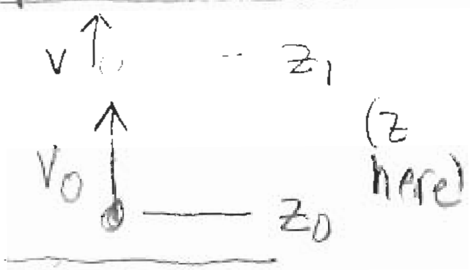
Escape velocity: $r_{max} = \infty$

Last Time... Work-Energy Theorem in 1-dimension

$$\frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2 = \int_{x_a}^{x_b} F(x)\,dx$$

Kinetic Energy $K = \frac{1}{2}mv^2$

Application #1 Vertical Trajectory



(z here)

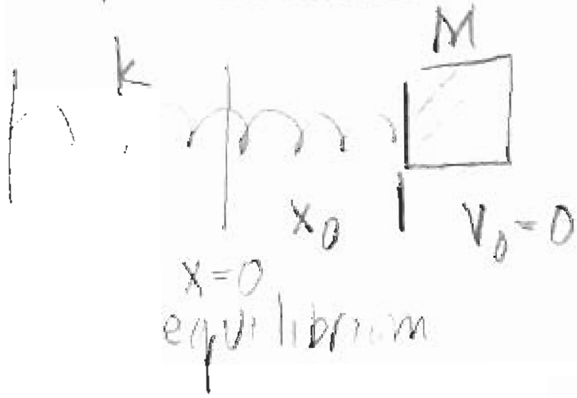
F = constant in

$$\frac{1}{2}m(v^2 - v_0^2) = -mg(z - z_0)$$

$$\boxed{v^2 = v_0^2 - 2g(z - z_0)}$$

time disappeared

Application #2 Spring $F = -kx$



$$\boxed{v^2 = v_0^2 + \frac{k}{M}(x_0^2 - x^2)}$$

- re-introduced time
- took $v_0 = 0$
- issue of dummy integration variables

$$\frac{dx}{dt} = -\sqrt{\frac{k}{m}(x_0^2 - x^2)}$$

$$\frac{dx}{\sqrt{x_0^2 - x^2}} = -\sqrt{\frac{k}{m}}\,dt \Rightarrow \int_{x_0}^{x} \frac{d\tilde{x}}{\sqrt{x_0^2 - \tilde{x}^2}} = -\sqrt{\frac{k}{m}}\int_0^t d\tilde{t}$$

$$\sin^{-1}\left(\frac{x}{x_0}\right) - \frac{\pi}{2} = -\sqrt{\frac{k}{m}}\,t$$

$$x = x_0 \sin\left(\frac{\pi}{2} - \sqrt{\frac{k}{m}}\,t\right)$$

$$x = x_0 \cos\left(\sqrt{\frac{k}{m}}\,t\right)$$

done.. $F \propto$ constant, distance $\rightarrow \frac{1}{\text{distance}^2}$

$$F(r) = -\frac{GM_e m}{r^2}$$

$$\frac{1}{2} m\left(v^2(r) - v_0^2\right) = -GM_e m \int_{R_e}^{r} \frac{d\tilde{r}}{\tilde{r}^2}$$

$$= -GM_e m\left(-\frac{1}{\tilde{r}}\right)\Bigg|_{R_e}^{r}$$

$$= -GM_e m\left(-\frac{1}{r} + \frac{1}{R_e}\right)$$

$$\frac{1}{2} m\left(v^2(r) - v_0^2\right) = -\underbrace{\frac{GM_e}{R_e^2}}_{\text{aka } g} \cdot m \cdot R_e\left(1 - \frac{R_e}{r}\right)$$

$$\boxed{v^2(r) = v_0^2 - 2gR_e\left(1 - \frac{R_e}{r}\right)}$$

$$v^2(\infty) = v_{0e}^2 - 2gR_e = 0 \quad \text{"escape velocity"}$$

$$v_{0e} = \sqrt{2gR_e} \simeq 11{,}000 \ \frac{\text{meters}}{\text{second}}$$

Connection with earlier formula...

$$r = R_e + (z - z_0) \qquad h \equiv z - z_0$$
$$= R_e + h$$

$$v^2(h) = v_0^2 - 2gR_e\left(1 - \frac{R_e}{R_e + h}\right)$$

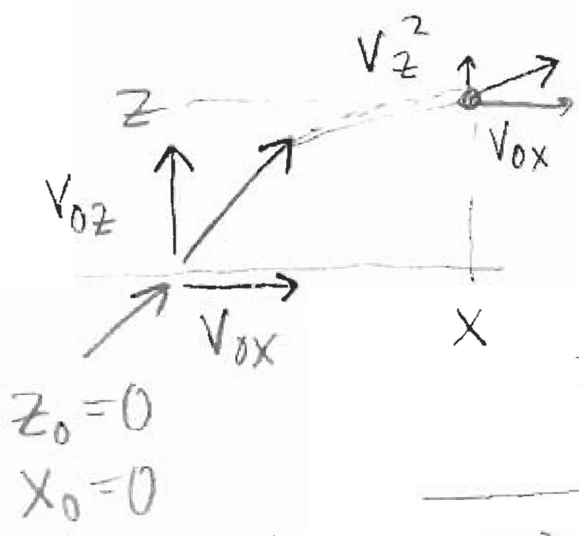
$$\frac{R_e}{R_e+h} \qquad \frac{1}{1 + h/R_e} \simeq 1 - \frac{h}{R_e} + \underbrace{\left(\frac{h}{R_e}\right)^2}_{\text{neglect}} + \dots$$

$$v^2(h) = v_0^2 - 2gR_e\left(1 - 1 + \frac{h}{R_e}\right)$$

$$\boxed{v^2(h) = v_0^2 - 2gh} \quad \text{as long as } h \ll R_e$$

In to the <u>second</u> dimension...

$\boxed{\text{rectilinear}}$



$$\frac{1}{2}mv_z^2 - \frac{1}{2}mv_{oz}^2 = \int_0^z F_z(\tilde{z})\,d\tilde{z} \qquad \begin{array}{c}\text{can't use}\\ z\dots\end{array}$$
$$= -mgz$$

$$\frac{1}{2}mv_x^2 - \frac{1}{2}mv_{ox}^2 = \int_0^x F_x(\tilde{x})\,d\tilde{x} = 0!$$

$$\frac{1}{2}m(v_x^2 + v_z^2) - \frac{1}{2}m(v_{ox}^2 + v_{oz}^2) = \int^z F_z(\tilde{z})\,d\tilde{z}$$
$$+ \int_0^x F_x(\tilde{x})\,d\tilde{x}$$

$$V_{escape}^2 = 2gR_e$$

$$V_{escape} = \sqrt{2g \cdot R_e} = \sqrt{2 \cdot 9.8 \cdot (6.4 \cdot 10^6)}$$

$$V_{escape} \simeq 11,000 \ \frac{\text{meters}}{\text{second}}$$

Near the surface of the earth:

$$r_{max} = R_e + h \qquad h \ll R_e \ , \ \frac{h}{R_e} \ll 1$$

$$V_0^2 = 2gR_e\left(1 - \frac{R_e}{R_e + h}\right)$$

$$= 2gR_e\left(\frac{R_e + h - R_e}{R_e + h}\right)$$

$$V_0^2 = \frac{2gR_e h}{R_e + h} = \frac{2g\cancel{R_e}h}{\cancel{R_e}} \cdot \frac{1}{1 + \frac{h}{R_e}}$$

$$V_0^2 = 2gh \cdot \frac{1}{1 + h/R_e}$$

$$\frac{1}{1+x} \stackrel{?}{=} 1 - x + x^2 - x^3 + x^4 - x^5$$

$$1 \stackrel{?}{=} (1+x)(1 - x + x^2 - x^3 + x^4 - x^5)$$

$$= 1 - x + x^2 - x^3 + x^4 - x^5$$
$$+ x - x^2 + x^3 - x^4 + x^5$$

$$= 1$$

when $h/R_e \ll 1$,

$$V_0^2 = 2gh\left(1 - \left(\frac{h}{R_e}\right) + \left(\frac{h}{R_e}\right)^2 - \left(\frac{h}{R_e}\right)^3 + \quad\right)$$

$$V_1^2 \simeq 2gh$$