Work + Energy

Impulse: $\vec{F}(t)$ $\quad \vec{P}_{final} - \vec{P}_i = \int \vec{F}(t)\,dt$

Work + Energy: $\vec{F}(\vec{r})$ ← most useful when no time dependence

Examples: Gravity: $\vec{F} = -G\frac{m_1 m_2}{r^2}\hat{r}$

Simple Harmonic Oscillator (1d) $\quad F = -kx$

In One-Dimension

$$m a = m\frac{d^2x}{dt^2} = F(x)$$

$$m\frac{dv}{dt} = F(x)$$

$$\underbrace{m\int_{x_a}^{x_b}\frac{dv}{dt}\,dx}_{} = \int_{x_a}^{x_b}F(x)\,dx$$

not an obvious integral — change variables:

$$dx = \left(\frac{dx}{dt}\right)dt = v\,dt$$

$$m\int_{x_a}^{x_b}\frac{dv}{dt}\,dx = m\int_{t(x_a)}^{t(x_b)}\frac{dv}{dt}v\,dt = \frac{1}{2}m\int_{t_a}^{t_b}\frac{d}{dt}(v^2)\,dt$$

$$\frac{d}{dt}\left(\frac{1}{2}v^2\right) = 2\cdot\frac{1}{2}$$

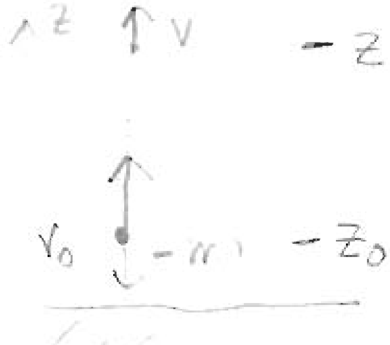
$$v_a = v(t_a) \qquad v_b = v(t_b)$$

so $$\frac{1}{2}m\left(v_b^2 - v_a^2\right) = \int_{x_a}^{x_b} F(x)\,dx$$

or $$\underbrace{\frac{1}{2}mv^2 - \frac{1}{2}mv_a^2 = \int_{x_a}^{x} F(x)\,dx}$$

consider second limit a var.

<u>Mass thrown upward</u> <u>z coordinates</u>



$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \int_{z_0}^{z}(-mg)\,dz$$

$$= -mgz\Big|_{z_0}^{z}$$

$$\frac{1}{2}m\left(v^2 - v_0^2\right) = -mg(z - z_0)$$

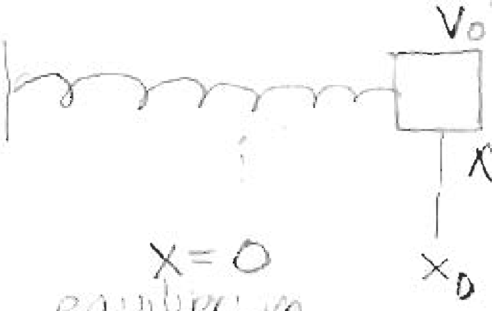
time not involved! $$\frac{1}{2}\left(v^2 - v_0^2\right) = -g(z - z_0)$$

<u>Top</u> of trajectory is when $v = 0$, $z = z_1$

$$-\frac{1}{2}v_0^2 = -g(z - z_0)$$

$$\boxed{z_1 = z_0 + \frac{v_0^2}{2g}} \qquad z \propto v^2$$

Simple Harmonic Oscillator



pull to here & release

equilibrium say

$$\frac{1}{2}Mv^2 - \frac{1}{2}Mv_0^2 = -k\int_{x_0}^{x} x\,dx$$

$$= -\frac{1}{2}kx^2 + \frac{1}{2}kx_0^2$$

$$v^2 - v_0^2 = -\frac{k}{M}x^2 + \frac{k}{M}x_0^2$$

decide to consider starting from rest

$x_0 = A\sin\omega t^{0} + B\cos\omega t = B$

$\dot{x}_0 = \omega A\cos\omega t - \omega B\sin\omega t = 0 = \omega A$

$x(t) = x_0\cos\omega t$

$$v = \frac{dx}{dt} = -\sqrt{\frac{k}{M}}\sqrt{x_0^2 - x^2}$$

$$\int_{x_0}^{x}\frac{dx}{\sqrt{x_0^2 - x^2}} = -\sqrt{\frac{k}{M}}\int_0^t dt = \sqrt{\frac{k}{M}}\,t$$

$$\sin^{-1}\left(\frac{x}{x_0}\right)\Bigg|_{x_0}^{x} = -\sqrt{\frac{k}{M}}\,t$$

$$\sin^{-1}\left(\frac{x}{x_0}\right) - \underbrace{\sin^{-1}(1)}_{\pi/2} = -\sqrt{\frac{k}{M}}\,t$$

$$\sin^{-1}\left(\frac{x}{x_0}\right) = -\sqrt{\frac{k}{M}}\,t + \frac{\pi}{2} \qquad \omega = \sqrt{\frac{k}{M}}$$

$$\frac{x}{x_0} = \sin\left(-\omega t + \frac{\pi}{2}\right)$$

$$x = x_0\sin\left(-\omega t + \frac{\pi}{2}\right) = x_0\cos(\omega t)$$