$$\vec{A}\cdot\vec{B} = \underbrace{(A_x\hat{\imath} + A_y\hat{\jmath})\cdot(B_x\hat{\imath} + B_y\hat{\jmath})}_{\text{dot product is distributive.}}$$

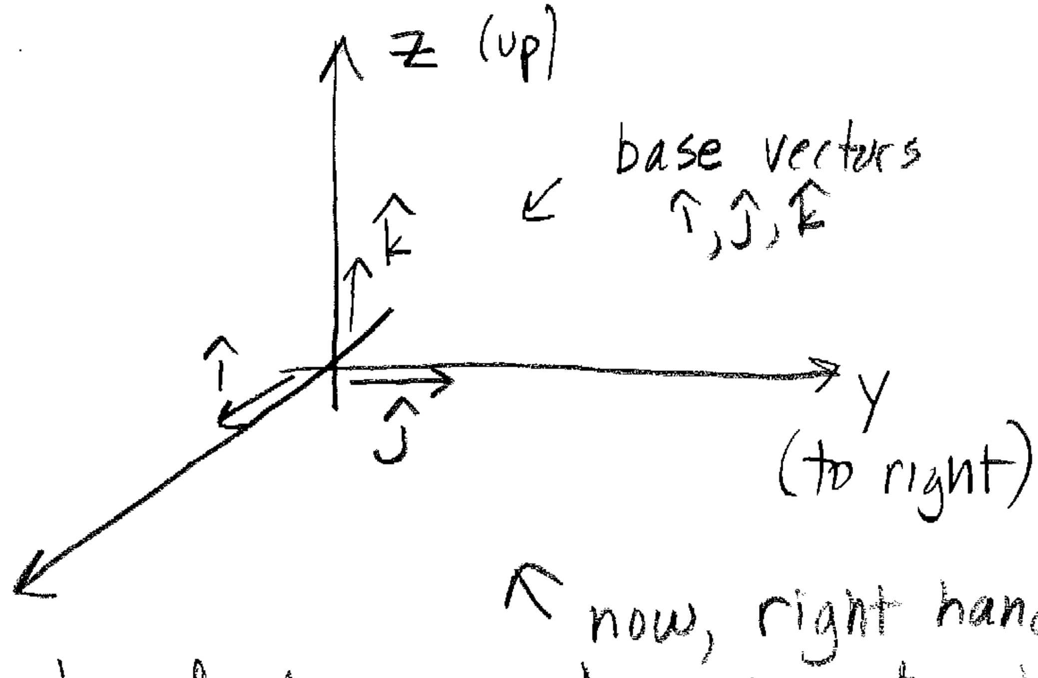
$$= A_xB_x\underbrace{\hat{\imath}\cdot\hat{\imath}}_{1} + A_xB_y\underbrace{\hat{\imath}\cdot\hat{\jmath}}_{0} + A_yB_x\underbrace{\hat{\jmath}\cdot\hat{\imath}}_{0} + A_yB_y\underbrace{\hat{\jmath}\cdot\hat{\jmath}}_{1}$$

$$\boxed{\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y}$$

note: $\vec{A}\cdot\vec{A} = A^2$!

Vector or "Cross" Product of Vectors

- used cos(angle between vectors)
  what about sin?
- Cross Product is innately three dimensional... into the third dimension.

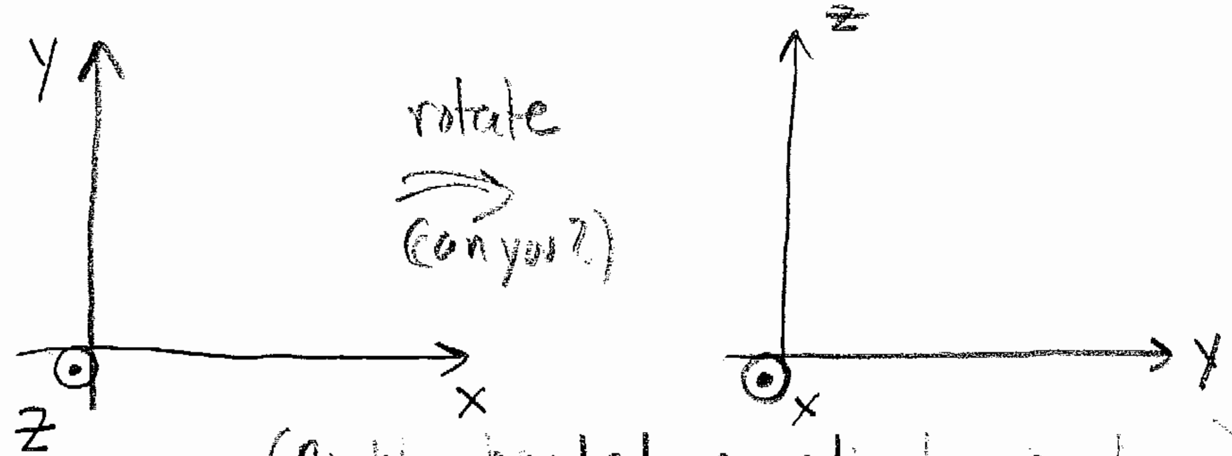


now, right hand rule gives thumb in z-direction.

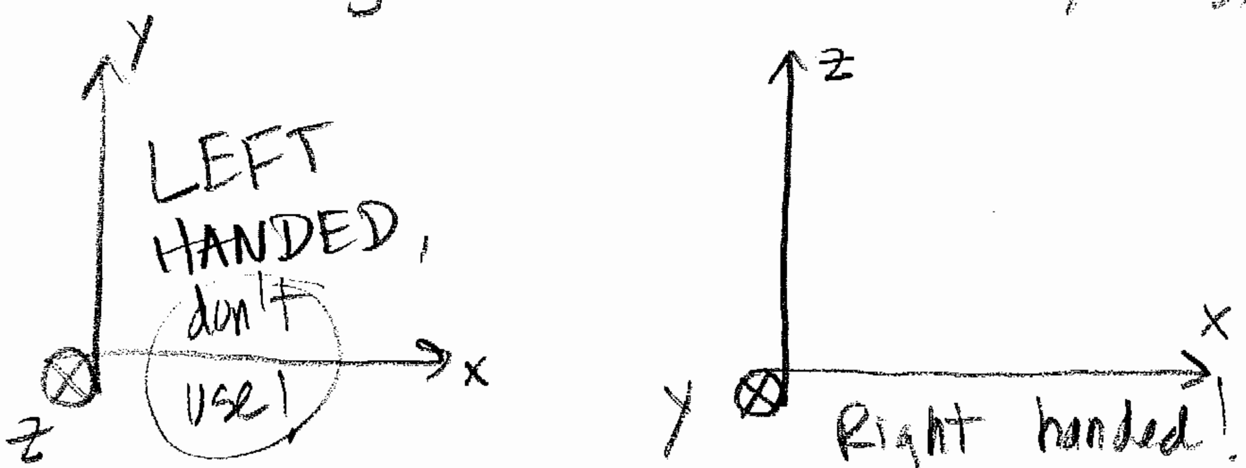
Another depiction

$\odot$ = vector coming out of page (like arrow tip)

$\otimes$ = vector going into page



(Right handed coordinate systems)

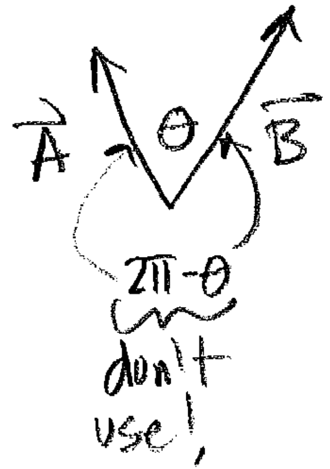


Cross-product generalizes these ideas to 2 (arbitrary) vectors...

$\vec{A}, \vec{B}$... magnitude of cross product

is $|\vec{A}||\vec{B}| \sin\theta$

smaller angle between.

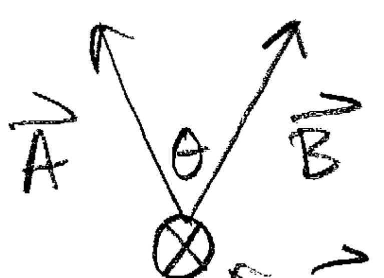


Magnitude of cross product $= |\vec{A}||\vec{B}|\sin\theta$

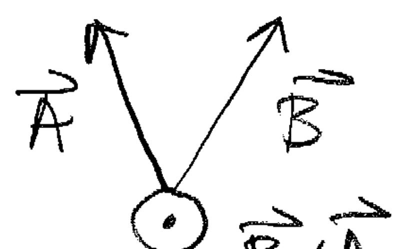
- Output of cross product is another vector! What is the direction?

RIGHT HAND RULE

$\vec{A}\times\vec{B}$:
- → align hand
- → fingers along first vector
- → curl fingers toward second vector
- → thumb points in direction of the vector product

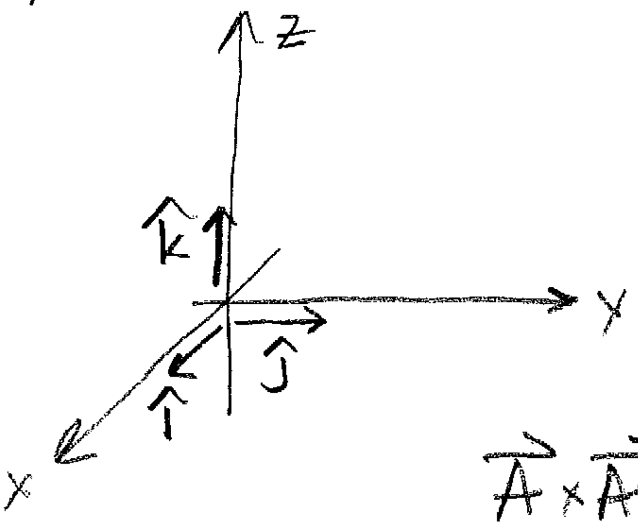


$\vec{A}\times\vec{B}$ points into page, magnitude $|\vec{A}||\vec{B}|\sin\theta$



$\vec{B}\times\vec{A} = -\vec{A}\times\vec{B}$

$\boxed{\text{NON-COMMUTATIVE}}$



cross product of base vectors...

$\underbrace{\hat{i} \times \hat{i}}$ = 0!

angle between is 0!

sin(angle) = 0

$\vec{A} \times \vec{A} = 0$ in general

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

$\hat{i} \times \hat{j} \rightarrow$ angle between is 90°

$\sin(90^\circ) = 1$

magnitude of each is 1

$$|\hat{i} \times \hat{j}| = 1 \cdot 1 \cdot \sin(90^\circ) = 1$$

$\underbrace{\text{direction}}$ $\rightarrow \hat{k}$

so.. $\hat{i} \times \hat{j} = \hat{k}$ $\quad \hat{j} \times \hat{k} = \hat{i}$ $\quad \hat{k} \times \hat{i} = \hat{j}$

$\hat{j} \times \hat{i} = -\hat{k}$ $\quad \hat{k} \times \hat{j} = -\hat{i}$ $\quad \hat{i} \times \hat{k} = -\hat{j}$

Take two vectors in x-y plane

$$\vec{A} = A_x \hat{i} + A_y \hat{j}$$

$$\vec{B} = B_x \hat{i} + B_y \hat{j}$$

$$(\vec{A} \times \vec{B}) = (A_x \hat{i} + A_y \hat{j}) \times (B_x \hat{i} + B_y \hat{j})$$

$$\vec{A}\times\vec{B} = A_xB_x\underbrace{\cancel{(\hat{\imath}\times\hat{\imath})}}_{0} + A_xB_y\overbrace{(\hat{\imath}\times\hat{\jmath})}^{\hat{k}} + A_yB_x\overbrace{(\hat{\jmath}\times\hat{\imath})}^{-\hat{k}} + A_yB_y\underbrace{\cancel{(\hat{\jmath}\times\hat{\jmath})}}_{0}$$

$$\underbrace{(\vec{A}\times\vec{B})}_{\text{in x-y plane}} = (A_xB_y - A_yB_x)\hat{k}$$

$\boxed{\vec{A}\times\vec{B} \text{ is in third dimension} \ldots z}$

$$(\vec{B}\times\vec{A}) = (B_xA_y - B_yA_x)\hat{k}$$
$$= -(\vec{A}\times\vec{B})$$

3-d Cross Product

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$
$$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$$

(could extend to more dimensions!)

use distributive property, $\hat{\imath}\times\hat{\jmath} = \hat{k}$ etc

$$\vec{A}\times\vec{B} = \underset{\text{"x"}}{\hat{\imath}}(A_yB_z - A_zB_y) + \underset{y}{\hat{\jmath}}(A_zB_x - A_xB_z) + \underset{z}{\hat{k}}(A_xB_y - A_yB_x)$$

order.. xyz is a "<u>permutation</u>" in "normal" order

"reversed order" → exchange two   xzy → -1 above

Determinant + Cross Product

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \equiv ad - bc \Rightarrow \text{involved in } \underline{\text{inverting}}\ 2\times 2\ \underline{\text{matrix}}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

answer is..
$$\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In two dimensions

$$\vec{A} \times \vec{B} = \hat{k} \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} = \hat{k}(A_x B_y - A_y B_x)$$

Three dimensions is tougher, but useful to know.