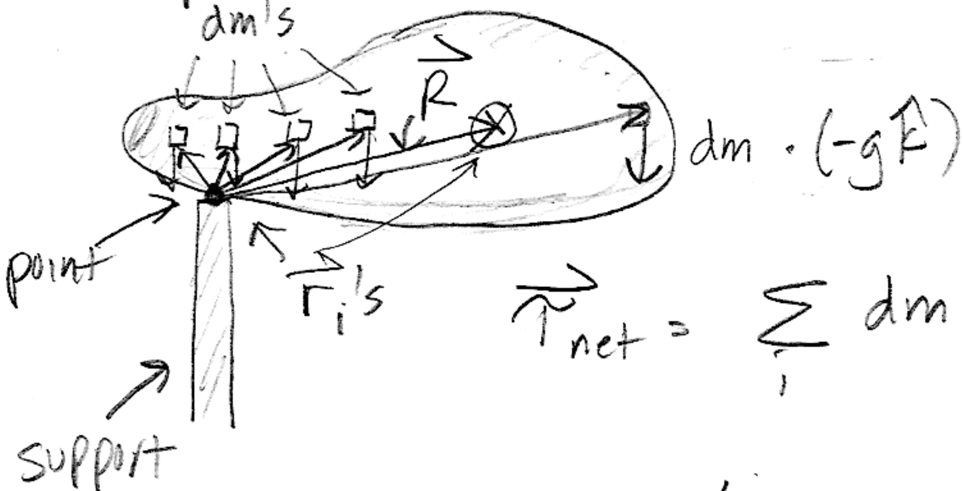
Torque on Center of Mass/Gravity



$$\vec{\tau}_{net} = \sum_i dm\, \vec{r}_i \times \underbrace{(-g\hat{k})}_{\text{constant}}$$

$$= \left(\sum_i dm\, \vec{r}_i\right) \times \underbrace{(-g\hat{k})}_{\text{since constant, can factor out}}$$

center of mass

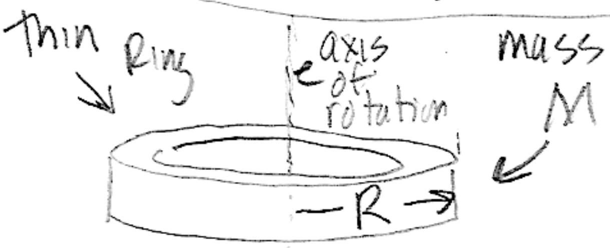
$$\vec{R} \equiv \frac{\sum_i dm\, \vec{r}_i}{\sum dm} = \frac{\sum_i dm\, \vec{r}_i}{M}$$

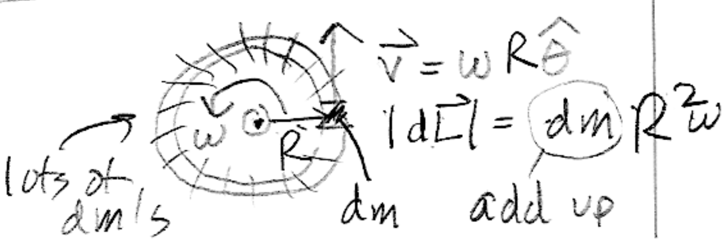
$$M\vec{R} = \sum_i dm\, \vec{r}_i$$

$$\boxed{\vec{\tau}_{Net} = M\vec{R} \times (-g\hat{k})}$$

(analyze, qualitatively, gyroscope).

Rotation Along a Fixed Axis Through Center of Mass





$$|\vec{L}| = \sum dm\, R^2 \omega$$

$$= \omega \sum dm\, R^2$$ (True for all rigid bodies rotating about fixed axis)

$$= \omega R^2 \cdot \sum dm$$ (true for ring)

$$|\vec{L}| = (MR^2) \times \omega$$

$$\equiv I \cdot \omega$$

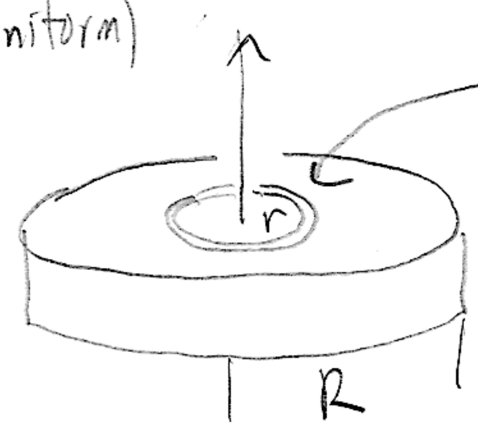
$$|\vec{p}| = m\,|\vec{v}|$$ ← think

I is like m, ω is like $|\vec{v}|$

I depends on the spatial distribution of mass in an object... at first, think of it as about the center of mass

Solid Disk (uniform)

① $I < MR^2$



$$dI = \underbrace{dm}\cdot r^2$$

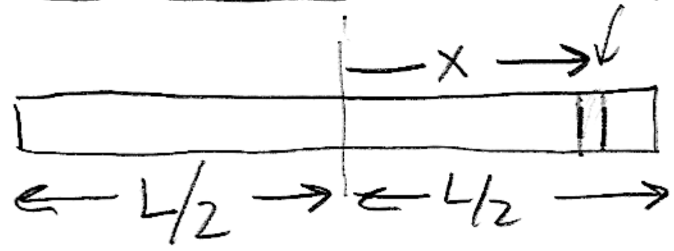
$$\frac{M}{\underbrace{\pi R^2}_{\text{total area}}} \cdot (\underbrace{2\pi r}_{\text{circumference of ring}}\ \underbrace{dr}_{\text{width of ring}})$$

total mass → M

$$\boxed{I = \frac{2M}{R^2}\int_0^R r^3\,dr = \frac{2M}{R^2}\frac{1}{4}R^4 = \frac{1}{2}MR^2}$$

Uniform Stick

$dm = \frac{M}{L} dx$



mass M

$$dI = x^2 \cdot \frac{M}{L} dx$$

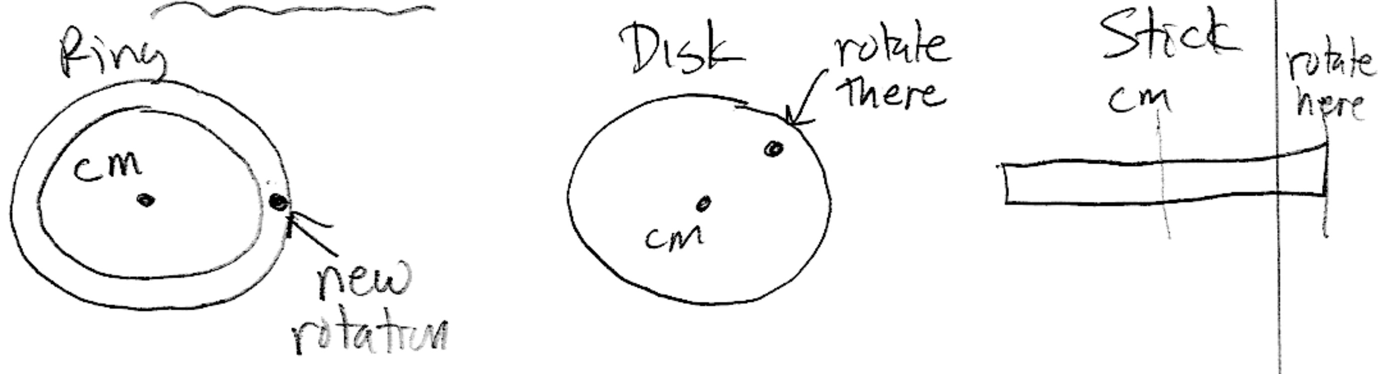
$$I = \frac{M}{L} \int_{-L/2}^{L/2} x^2 dx = \frac{M}{L} \frac{1}{3} x^3 \Big|_{-L/2}^{L/2}$$

$$= \frac{M}{L} \cdot \frac{1}{3} \cdot \underbrace{\left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3 \right]}_{\frac{2}{8} L^3}$$

$$\boxed{I = \frac{1}{12} M L^2}$$

Parallel Axis Theorem

What if one rotates about a new axis, parallel to that for which I_o c.m. about center of mass was computed, but translated:



You kind of already know the answer.

Trick is to evaluate I in two steps

$$\vec{r}_i = \vec{R}_\perp + \vec{\rho}_i$$

meaning: $\sum dm_i \vec{\rho}_i = 0$

$\vec{r}_i$: from new axis to dab of mass dm

$\vec{R}_\perp$: from new axis to center of mass

$\vec{\rho}_i$: from center of mass to dab of mass

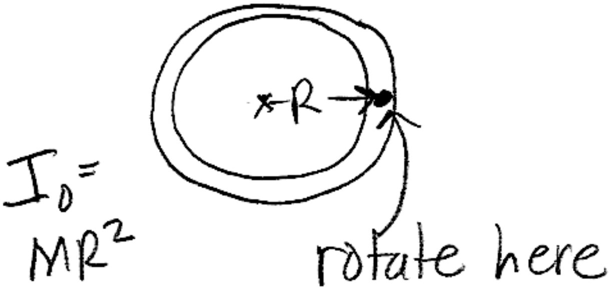
(in 3-d $\vec{R}_\perp$ is also $\perp$ to axis of rotation.. not a big deal here, where $|\vec{R}_\perp| = \ell$ all that matters)

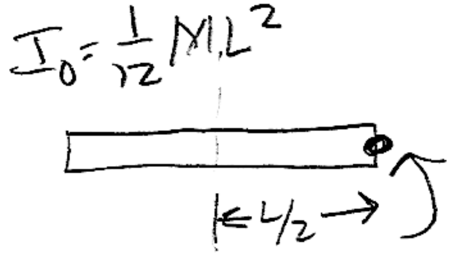
$$I = \sum_i dm_i |\vec{r}_i|^2 = \sum_i dm_i (\vec{R}_\perp + \vec{\rho}_i)^2$$

$$= \sum_i dm_i \left(\ell^2 + 2\vec{\rho}_i \cdot \vec{R}_\perp + |\vec{\rho}_i|^2 \right)$$

but $\sum dm_i \vec{\rho}_i \cdot \vec{R}_\perp = \vec{R}_\perp \cdot \underbrace{\sum dm \, \vec{\rho}_i}_{\vec{0}}$ since measuring $\vec{\rho}_i$ w/r to center of mass

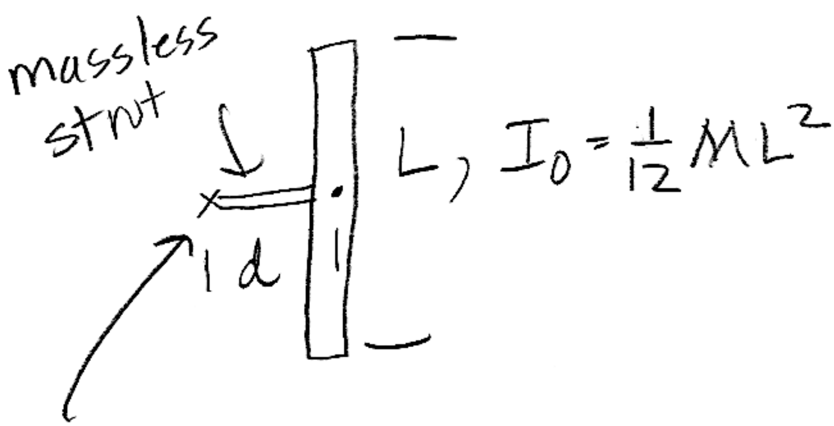
$$\boxed{I = M\ell^2 + I_0}$$





$$I = I_0 + MR^2$$
$$= MR^2 + MR^2$$
$$\boxed{I = 2MR^2}$$

$$I = I_0 + \left(\frac{L}{2}\right)^2$$
$$= \left(\frac{1}{12} + \frac{1}{4}\right)ML^2$$
$$= \frac{4}{12}ML^2$$
$$\boxed{I = \frac{1}{3}ML^2}$$



$$I = Md^2 + \frac{1}{12}ML^2$$