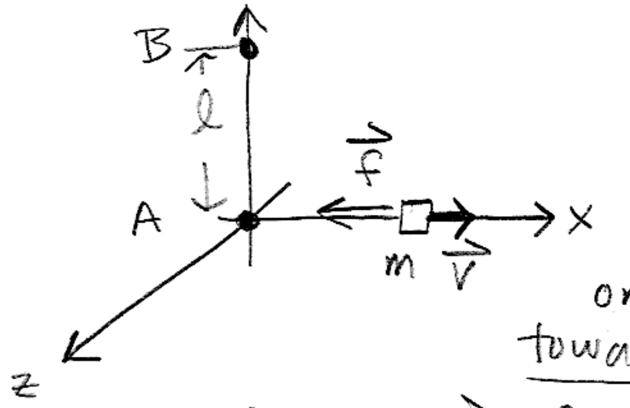
Vector Nature of $\vec{L}$, $\vec{\tau}$

Recall block moving along x axis



- with respect to point A:

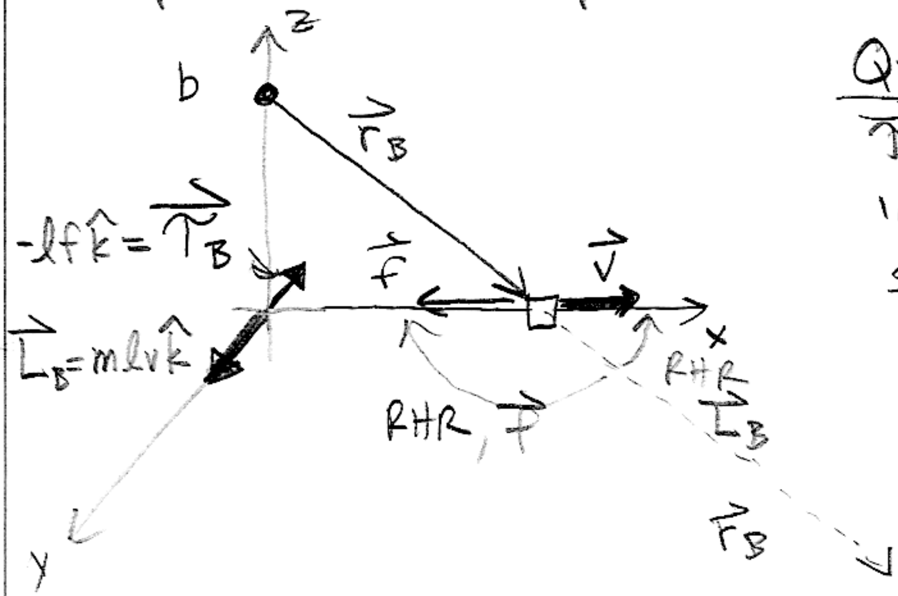
$$\vec{L}_A = 0$$

- now introduce force $\vec{f}$ on m, pointing back toward origin.

Either $\vec{r}_\perp$ for $\vec{f}$, or $\vec{f}_\perp = 0$.
or, $\vec{r} \times \vec{f} = 0$. So for $\vec{f}$:

$$\vec{\tau}_A = 0 \Rightarrow \frac{d\vec{L}_A}{dt} = 0, \quad \vec{L}_A = 0 \text{ forever.}$$

HOWEVER, using point A results in consistent equations, but misses the description of the acceleration of the block entirely! Look at problem from point B:



Qualitatively,
$\vec{\tau}_B$ causes decrease in $\vec{L}_B$ (over time), since $\vec{L}_B$ in $+\hat{k}$ direction
$\vec{\tau}_B$ in $-\hat{k}$ direction

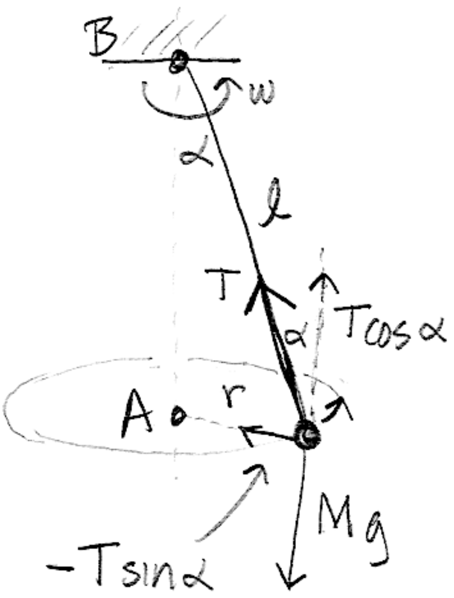
Quantitatively

$$\vec{\tau}_B = \frac{d\vec{L}_B}{dt}$$

$$-\ell f\hat{k} = \frac{d}{dt}(m\ell v\hat{k})$$

Same as N2 $\Rightarrow$ $-f = m\frac{dv}{dt} = ma$

Conical Pendulum $\Rightarrow$ more interesting...



recall: $T\cos\alpha = Mg$

$$-M\frac{v^2}{r} = -M\frac{r^2\omega^2}{r} = -T\sin\alpha$$

or $Mr\omega^2 = T\sin\alpha$

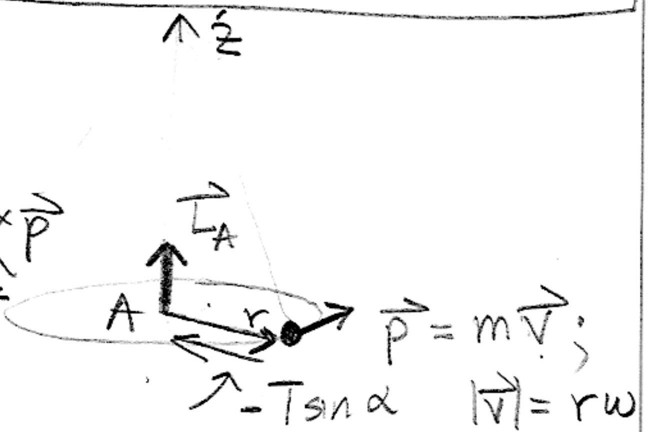
from "rectilinear analysis"

ratio: $\tan\alpha = \frac{r}{g}\omega^2 = \frac{\ell\sin\alpha}{g}\omega^2, \ \omega^2 = \frac{g}{\ell\cos\alpha}$

$\boxed{\text{Analyze about point A}}$

$\vec{L}_A$:

$\vec{L}_A = \vec{r}\times\vec{p}$
$= mr^2\omega\hat{k}$



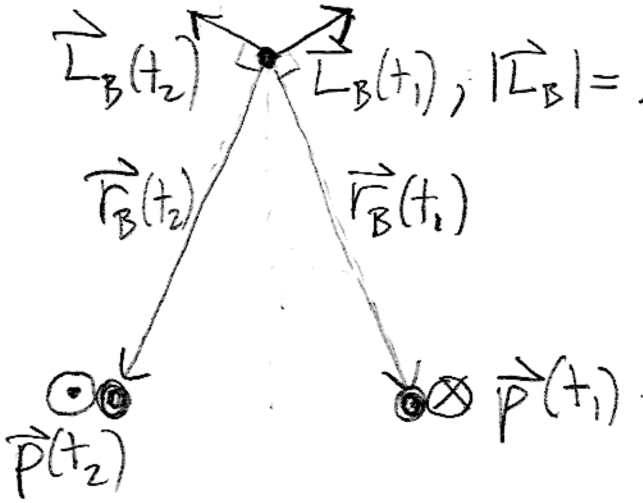
$\vec{\tau}_A$: NET force is only $-T\sin\alpha$ toward origin. (Mg + $T\cos\alpha$ cancel).

$$\vec{\tau}_A = \vec{r}\times(-T\sin\alpha\,\hat{r}) = 0.$$

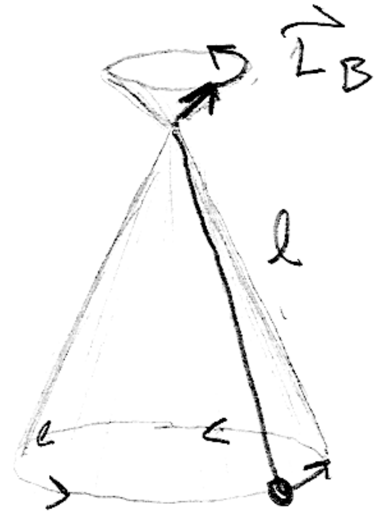
Conclude... $\frac{d\vec{L}_A}{dt} = 0$, $\vec{L}_A = \text{constant}$
$= mr^2\omega\hat{k}$

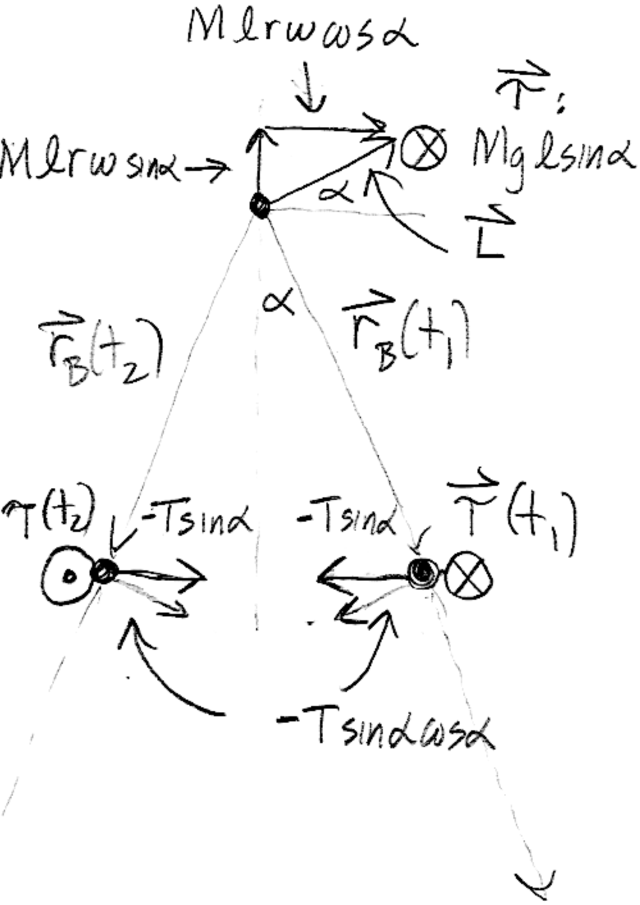
Nothing so far in the torque description gives $Mr\omega^2 = T\sin\alpha$, which relates the key quantities...

Analyze w/r to point B:



$|\vec{L}_B| = \ell M r \omega$





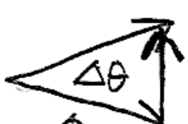
$\vec{L}_B$ itself moves around on a small cone, "tracking" the bob.

$$|\vec{\tau}(\text{all time})| = \ell T\sin\alpha\cos\alpha$$

$$T\cos\alpha = Mg$$

$$= Mg\ell\sin\alpha$$

Look from top:



$$|\Delta\vec{L}| = (Mg\ell\sin\alpha)\cdot\Delta t \quad \left(\text{since } \vec{\tau} = \frac{d\vec{L}}{dt}\right)$$

projection of $\vec{L}$, Magnitude $= M\ell r\omega\cos\alpha$

but: $\dfrac{|\Delta \vec{L}|}{\text{projection of } \vec{L}} = \Delta\theta = \dfrac{\cancel{M} g \ell \sin\alpha \, \Delta t}{\cancel{M} \ell r \omega \cos\alpha}$

or $\omega = \dfrac{\Delta\theta}{\Delta t} = \dfrac{g}{r\omega} \tan\alpha$

$$\omega^2 = \frac{g}{r} \tan\alpha \qquad r = \ell \sin\alpha$$

$$\boxed{\omega^2 = \frac{g}{\ell \sin\alpha} \cdot \frac{\sin\alpha}{\cos\alpha} = \frac{g}{\ell \cos\alpha}}$$

same as rectilinear