Work + Energy

Impulse: $\vec{F}(t)$, $\vec{P}_{final} - \vec{P}_{initial} = \int \vec{F}(t)\,dt$

Work + Energy: $\vec{F}(\vec{r})$

most useful when no time dependence

Examples: Gravity: $\vec{F} = -G\frac{m_1 m_2}{r^2}\hat{r} \xrightarrow[\text{of earth}]{\text{surface}} -m\vec{g}$

Simple Harmonic Oscillator (1d) $F = -kx$

In One-Dimension:

$$ma = m\frac{d^2x}{dt^2} = F(x)$$

$$m\frac{dv}{dt} = F(x)$$

$$\underbrace{m\int_{x_a}^{x_b}\frac{dv}{dt}\,dx} = \int_{x_a}^{x_b} F(x)\,dx$$

not an obvious integral ... change variables: $dx = \left(\frac{dx}{dt}\right)dt = v\,dt$

$$m\int_{x_a}^{x_b}\frac{dv}{dt}\,dx = m\int_{t(x_a)}^{t(x_b)}\frac{dv}{dt}\,v\,dt = \frac{1}{2}m\int_{t_a}^{t_b}\frac{d}{dt}(v^2)\,dt$$

$$\frac{d}{dt}\left(\frac{1}{2}v^2\right) = 2\times\frac{1}{2}\cdot v\cdot\frac{dv}{dt} = v\frac{dv}{dt}$$

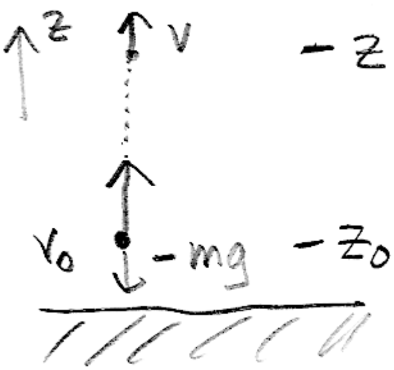
$$V_a = V(t_a) \qquad V_b = V(t_b)$$

so $$\frac{1}{2}m\left(V_b^2 - V_a^2\right) = \int_{x_a}^{x_b} F(x)\,dx$$

or $$\underbrace{\frac{1}{2}mv^2 - \frac{1}{2}mv_a^2 = \int_{x_a}^{x} F(x)\,dx}$$

consider second limit a variable.

Mass thrown upward. z coordinates



$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \int_{z_0}^{z}(-mg)\,dz$$

$$= -mgz\Big|_{z_0}^{z}$$

$$\frac{1}{2}m(v^2 - v_0^2) = -mg(z - z_0)$$

time not involved!

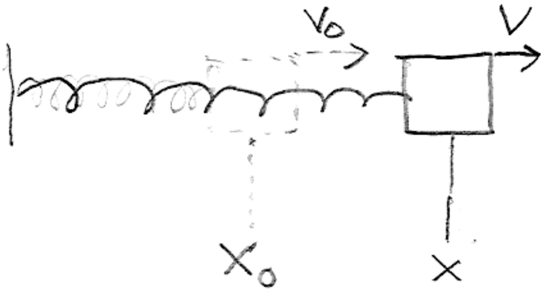
$$\frac{1}{2}(v^2 - v_0^2) = -g(z - z_0)$$

Top of trajectory is when ... $v = 0$, $z = z_1$

$$-\frac{1}{2}v_0^2 = -g(z - z_0)$$

$$\boxed{z_1 = z_0 + \frac{v_0^2}{2g}} \qquad z \propto v^2$$

Simple Harmonic Oscillator



$$\frac{1}{2}Mv^2 - \frac{1}{2}Mv_0^2 = -k\int_{x_0}^{x} x\,dx$$

$$= -\frac{1}{2}kx^2 + \frac{1}{2}kx_0^2$$

$$v^2 - v_0^2 = -\frac{k}{M}x^2 + \frac{k}{M}x_0^2$$

decide to consider starting from rest

$$x_0 = A\cancel{\sin\omega t}^{\,0} + B\cos\omega t^{\,1} = B$$

$$\dot{x}_0 = \omega A\cos\omega t - \omega B\sin\omega t = 0 = \omega A$$

$$x(t) = x_0\cos\omega t$$

$$v = \frac{dx}{dt} = \sqrt{\frac{k}{M}}\sqrt{x_0^2 - x^2}$$

$$\int_{x_0}^{x}\frac{dx}{\sqrt{x_0^2 - x^2}} = \sqrt{\frac{k}{M}}\int_0^t dt = \sqrt{\frac{k}{M}}\,t$$

$$\sin^{-1}\left(\frac{x}{x_0}\right)\Bigg|_{x_0}^{x} = \sqrt{\frac{k}{M}}\,t$$

$$\sin^{-1}\left(\frac{x}{x_0}\right) - \underbrace{\sin^{-1}(1)}_{\pi/2} = \sqrt{\frac{k}{M}}\,t$$

$$\sin^{-1}\left(\frac{x}{x_0}\right) = \sqrt{\frac{k}{M}}\,t + \frac{\pi}{2} \qquad \omega \equiv \sqrt{\frac{k}{M}}$$

$$\frac{x}{x_0} = \sin\left(\omega t + \frac{\pi}{2}\right)$$

$$\boxed{x = x_0\sin\left(\omega t + \frac{\pi}{2}\right) = x_0\cos(\omega t)}$$

Work Energy Theorem in One Dimension

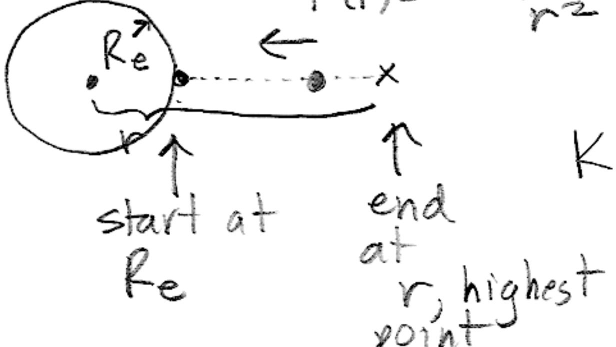
$$\frac{1}{2}mv^2 \equiv \text{Kinetic Energy} = K$$

units : $kg \cdot \frac{m^2}{s^2} \equiv$ Joule

$$\frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2 = K_b - K_a = \underbrace{\int_{x_a}^{x_b} F(x)\,dx}_{\text{work } W_{ba}}$$

$$K_b - K_a = W_{ba} \quad \text{"Work-Energy Theorem"}$$

$$F(r) = -\frac{GM_e m}{r^2}$$ (− means back toward origin)



$$K(r) - K(R_e) = -GM_e m \int_{R_e}^{r} \frac{dr}{r^2}$$

$$= -GM_e m \left(-\frac{1}{r} \right) \Big|_{R_e}^{r}$$

$$\frac{1}{2}\cancel{m}\left(v^2(r) - v_0^2\right) = GM_e \cancel{m}\left(\frac{1}{r} - \frac{1}{R_e}\right)$$

$\rightarrow v(r) = 0$ when at highest point, $r = r_{max}$

$$v_0^2 = 2M_eG\left(\frac{1}{R_e} - \frac{1}{r_{max}}\right)$$

$$= 2\underbrace{\left(\frac{M_eG}{R_e^2}\right)}_{g} R_e \left(1 - \frac{R_e}{r_{max}}\right)$$

$$\boxed{v_0^2 = 2gR_e\left(1 - \frac{R_e}{r_{max}}\right)}$$

Escape velocity : $r_{max} = \infty$

$$V_{escape}^2 = 2gR_e$$

$$V_{escape} = \sqrt{2g \cdot R_e} = \sqrt{2 \cdot 9.8 \cdot (6.4 \cdot 10^6)}$$

$$\boxed{V_{escape} \simeq 11,000 \ \frac{\text{meters}}{\text{second}}}$$

Near the surface of the earth:

$$r_{max} = R_e + h \qquad h \ll R_e \ , \ \frac{h}{R_e} \ll 1$$

$$V_0^2 = 2gR_e\left(1 - \frac{R_e}{R_e+h}\right)$$

$$= 2gR_e\left(\frac{R_e+h-R_e}{R_e+h}\right)$$

$$V_0^2 = \frac{2gR_eh}{R_e+h} = \frac{2g\cancel{R_e}h}{\cancel{R_e}} \cdot \frac{1}{1+\frac{h}{R_e}}$$

$$V_0^2 = 2gh \cdot \left(\frac{1}{1+h/R_e}\right)$$

$$\frac{1}{1+x} \stackrel{?}{=} 1 - x + x^2 - x^3 + x^4 - x^5 \ldots$$

$$1 \stackrel{?}{=} (1+x)(1 - x + x^2 - x^3 + x^4 - x^5 \ldots)$$

$$= 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots$$
$$+ x - x^2 + x^3 - x^4 + x^5 + \ldots$$

$$= 1.$$

when $(h/R_e) \ll 1$,

$$V_0^2 = 2gh\left(1 - \left(\frac{h}{R_e}\right) + \left(\frac{h}{R_e}\right)^2 - \left(\frac{h}{R_e}\right)^3 + \ldots\right)$$

$$V_0^2 \simeq 2gh$$