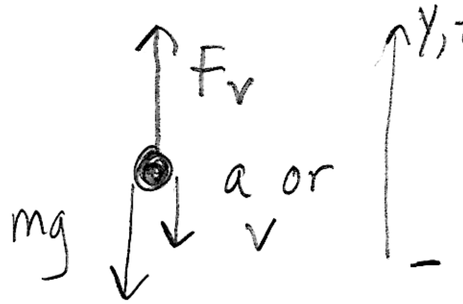
Problem #7



a has negative component! v too. so does mg

Project on y-axis:

$$-mg + F_v = -ma$$

$$-mg + F_v = -m\frac{dv}{dt}$$

$$F_v = -C\cdot(-v) = Cv$$

so, $$-mg + Cv = -m\frac{dv}{dt}$$

Separate variables

$$\int_{v_0=0}^{v} \frac{m\,dv}{mg - Cv} = \int_0^t dt$$

$$-\frac{m}{c}\ln(mg - Cv)\Big|_0^v = t$$

$$\ln\frac{mg - Cv}{mg} = -\frac{c}{m}t$$

$$1 - \frac{Cv}{mg} = e^{-\frac{c}{m}t}$$

$$\boxed{v(t) = \frac{mg}{c}\cdot\left(1 - e^{-\frac{c}{m}t}\right)}$$

$$\tilde{v} = v + v_f$$ constant, choose for convenience

$$\frac{d\tilde{v}}{dt} = \frac{dv}{dt}$$

$$-mg + C(\tilde{v} + v_f) = -m\frac{d\tilde{v}}{dt}$$

choose $-mg + Cv_f = 0$

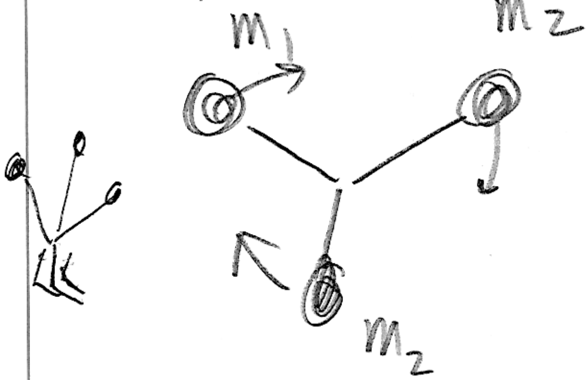
$$\boxed{v_f = \frac{mg}{C}}$$

Then: $$C\tilde{v} = -m\frac{d\tilde{v}}{dt}$$

or $$\boxed{\frac{d\tilde{v}}{dt} = -\frac{c}{m}\tilde{v}}$$

$$\boxed{\sum_i \frac{d\vec{p}_i}{dt} = \sum_j \vec{F}_{ext,j}}$$ can neglect internal forces.

Flying "bola"



$$\frac{d}{dt}(\vec{p}_1+\vec{p}_2+\vec{p}_3) = m_1\vec{g} + m_2\vec{g} + m_3\vec{g}$$

neglect string!

$$\vec{P} \equiv \vec{p}_1+\vec{p}_2+\vec{p}_3 \qquad \frac{d\vec{P}}{dt} = \underbrace{(m_1+m_2+m_3)}_{M}\vec{g} = M\vec{g}$$

motion ($\vec{p}_1(t)$) of individual is complicated but → motion of $\vec{P}$ sum is that of a single particle, mass M.

Center of mass

Trajectory of precisely what follows that of something with mass M?

Find an $\vec{R}$ such that

$$\vec{F} = M\ddot{\vec{R}} = \frac{d}{dt}\sum \vec{p}_i = \underbrace{\frac{d}{dt}\sum m_i\vec{v}_i}_{\text{assume } \frac{dm_i}{dt}=0} = \sum m_i \ddot{\vec{r}}_i$$

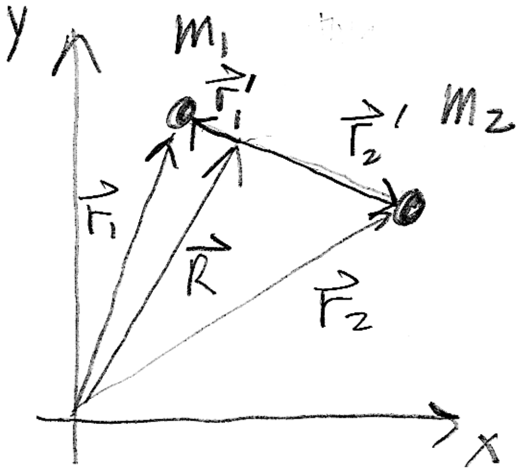
$$M\ddot{\vec{R}} = \sum m_i \ddot{\vec{r}}_i$$

$$\vec{R} \equiv \frac{1}{M}\sum m_i \vec{r}_i = \frac{\sum m_i \vec{r}_i}{\sum m_i}$$

"center of mass"

"m-weighted mean position"

Example: baton: two masses joined by thin rod, infinitesimal mass, length ℓ



center of mass is:
1) on a line beween.
2) closer to the heavier mass.

Formula: $\vec{R} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$

- $\vec{r}$ to bigger one has bigger coefficient.
- $\lim\limits_{m_1\to\infty} \vec{R} = \vec{r}_1$, $\lim\limits_{m_2\to\infty}\vec{R} = \vec{r}_2$

$$\vec{r}_1' = \vec{r}_1 - \vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_1 - m_1\vec{r}_1 - m_2\vec{r}_2}{m_1+m_2} = \frac{m_2}{m_1+m_2}(\vec{r}_1 - \vec{r}_2)$$

$$\vec{r}_2' = \vec{r}_2 - \vec{R} = \frac{m_1\vec{r}_2 + \cancel{m_2\vec{r}_2} - m_1\vec{r}_1 - \cancel{m_2\vec{r}_2}}{m_1+m_2} = \frac{-m_1}{m_1+m_2}(\vec{r}_1 - \vec{r}_2)$$

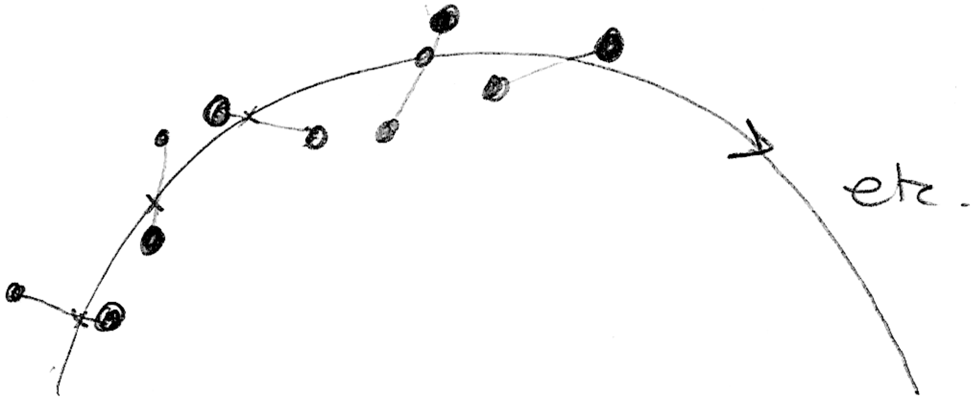
both $\propto \vec{r}_1 - \vec{r}_2$, so $\vec{R}$ must be on line between.

$$|\vec{r}_1{}'| = \frac{m_2}{m_1+m_2}|\vec{r}_1-\vec{r}_2| = \frac{m_2}{m_1+m_2}\,l, \propto m_2$$

$$|\vec{r}_2{}'| = \frac{m_1}{m_1+m_2}|\vec{r}_1-\vec{r}_2| = \frac{m_1}{m_1+m_2}\,l, \propto m_1$$

these are proportional to the other, so the $|\vec{r}_i|$ to the big mass will be small, and vice versa.

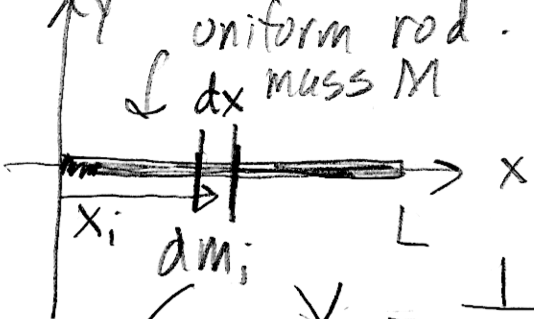
When thrown, the center of mass follows the parabolic trajectory.



Go to 65 earth-moon

Continuous mass distributions

uniform rod. mass M ← "obviously"



$$X = \frac{L}{2} \quad Y = 0 \quad Z = 0$$

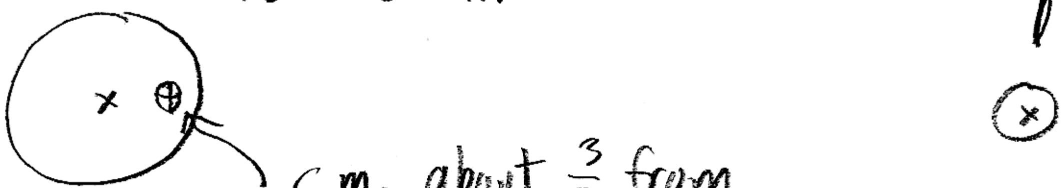
$$X = \frac{1}{M}\cdot\sum x_i\, dm_i \Rightarrow \frac{1}{M}\int_0^L x\frac{M}{L}\cdot dx$$

$$dm_i = \frac{M}{L}\cdot dx$$

$$= \frac{1}{L}\int_0^L x\,dx = \frac{1}{L}\frac{1}{2}x^2\Big|_0^L = \underline{\underline{\frac{1}{2}L}}$$

The Earth and the Moon

$\ell = 3.84 \cdot 10^8\ m \longrightarrow$



c.m. about $\frac{3}{4}$ from earth center to surface

Earth
$M_e = 5.98 \cdot 10^{24}\ kg$
$R_e = 6.37 \cdot 10^6\ m$

Moon
$M_m = 7.34 \cdot 10^{22}\ kg$
$R_m = 1.74 \cdot 10^6\ m$

Center of mass:

$$\vec{R}_{cm} = \frac{M_e \cdot \vec{r}_e + M_m \vec{r}_m}{M_e + M_m}$$

(as if all mass concentrated at center when symmetric about center)

Choose smart origin: on center of earth!

$$\vec{R}_{cm} = \frac{M_m}{M_e + M_m} \vec{r}_m$$

($\vec{r}_m \rightarrow$ points from earth to moon)

$$= \frac{M_m / M_e}{1 + M_m / M_e} \cdot \vec{r}_m$$

($\vec{r}_m = \ell \hat{x}$, $\ell \to 3.84 \cdot 10^8\ m$)

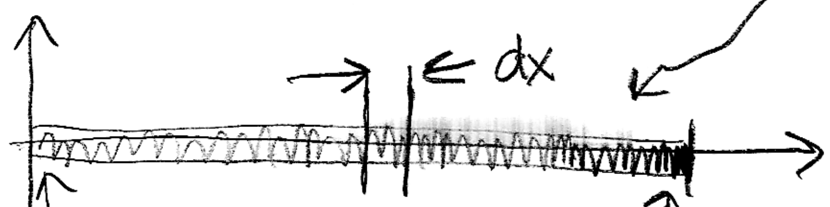
$$\frac{M_m}{M_e} = \frac{7.34 \cdot 10^{22}}{5.98 \cdot 10^{24}} \frac{kg}{kg} = \frac{1}{81.5}$$

$$\vec{R}_{cm} = \frac{1/81.5}{1 + 1/81.5} \times 3.84 \cdot 10^8\ m \cdot \hat{x} = \frac{1}{82.5} \cdot 3.84 \cdot 10^8\ m \cdot \hat{x}$$

$$= 4.65 \cdot 10^6\ m\ \hat{x}$$

$$\frac{|\vec{R}_{cm}|}{R_e} = \frac{4.65 \cdot 10^6\ m}{6.37 \cdot 10^6\ m} = 0.731$$

Non-uniform density



question: how much mass is in the slice dx?

$$dm = \lambda(x)\,dx$$

Two steps: "Normalization"

(1) $$\int dm = \int_0^L \lambda(x)\,dx = M$$

This step specifies constants...

(2) $$x_{cm} = \frac{1}{M}\int x\,dm = \frac{1}{M}\int_0^L x\lambda(x)\,dx$$

example: $\lambda(x) \propto x$ (linear increase)

$$= \lambda_0 x$$

(1) Normalization: $$\int_0^L \lambda_0 x\,dx = \lambda_0 \frac{x^2}{2}\Big|_0^L = \frac{\lambda_0 L^2}{2} = M$$

$$\lambda_0 = \frac{2M}{L^2}$$

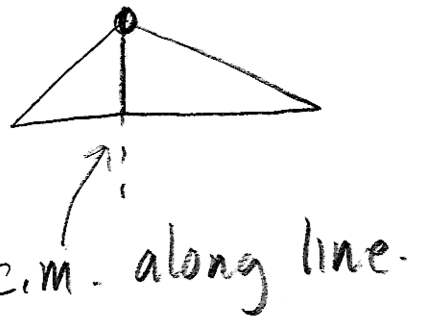
$$\lambda(x) = 2\frac{M}{L}\cdot\frac{x}{L}$$

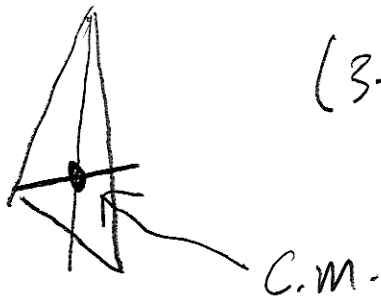
(2) $$x_{cm} = \frac{1}{M}\int_0^L x\,\lambda(x)\,dx = \frac{1}{M}\int_0^L x\cdot 2\frac{M}{L}\frac{x}{L}\,dx$$

$$= \frac{2}{L^2}\int_0^L x^2\,dx = \frac{2}{3L^2}\,x^3\Big|_0^L$$

$$\boxed{x_{cm} = \frac{2}{3}L}$$

2-d: best way is to hang + intersect





(3-d

Conservation of Momentum

System: $\vec{P} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \cdots$

then $\frac{d\vec{P}}{dt} = \sum \vec{F}_{ext,i}$

what happens when $\sum \vec{F}_{ext,i} = 0$

$\frac{d\vec{P}}{dt} = 0$ } may happen for only 1 component