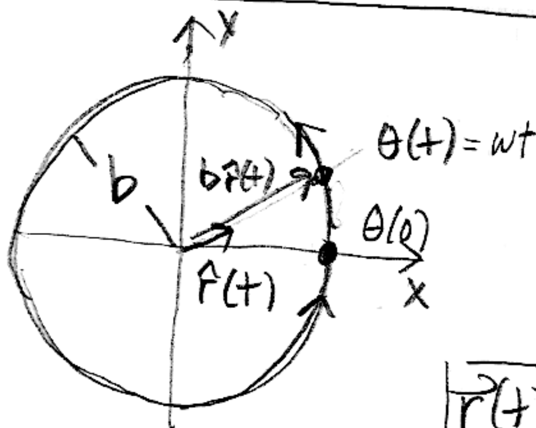
Acceleration in Polar Coordinates



$\vec r(t) = b\,\hat r(t)$ where is θ?

$\hat r = \cos\theta\,\hat\imath + \sin\theta\,\hat\jmath$

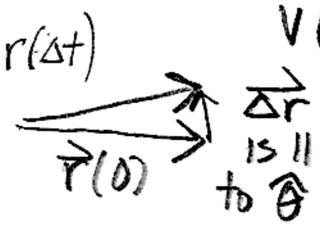
$= \cos(\omega t)\,\hat\imath + \sin(\omega t)\,\hat\jmath$

or

$$\boxed{\vec r(t) = b\cos(\omega t)\,\hat\imath + b\sin(\omega t)\,\hat\jmath}$$

point going around a circle of radius b

now, get $\vec v(t) = \frac{d\vec r}{dt} = \frac{d}{dt}\left(b\,\hat r(t)\right) = \cancel{\frac{db}{dt}}^{\,0}\,\hat r + b\frac{d\hat r}{dt}$



$$\vec v(t) = b\frac{d\hat r}{dt} = b\,\dot\theta\,\hat\theta = b\left(\frac{d(\omega t)}{dt}\right)\hat\theta = b\omega\,\hat\theta$$

another way:

$$\vec r(t) = b\cos(\omega t)\,\hat\imath + b\sin(\omega t)\,\hat\jmath$$

$$\frac{d\vec r}{dt} = -(\omega b)\sin(\omega t)\,\hat\imath + (\omega b)\cos(\omega t)\,\hat\jmath$$

$$= b\omega\left[-\sin(\theta)\,\hat\imath + \cos(\theta)\,\hat\jmath\right]$$

$$= b\omega\,\hat\theta$$

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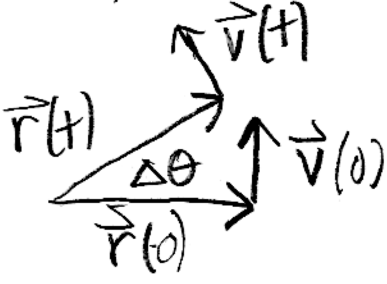
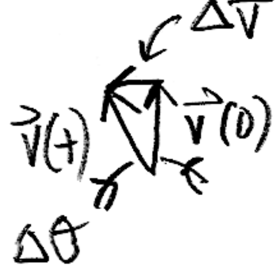
now, to acceleration....



$\Delta\vec v$ is toward the origin!



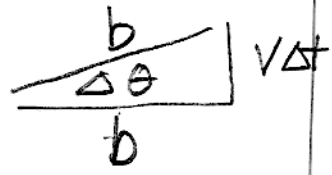
$\Delta\vec v \simeq |\vec v|\,\Delta\theta\,(-\hat r)$

$$\frac{\Delta\vec v}{\Delta t} \simeq \underbrace{|\vec v|}_{b\omega}\,\underbrace{\frac{\Delta\theta}{\Delta t}}_{\omega}(-\hat r)$$

circular motion, constant speed $\boxed{\vec a = -b\omega^2\,\hat r}$

A body going with constant speed on a circular path of radius b is being accelerated toward the origin of the circle with acceleration:

$$b\omega^2 \quad \text{— note} \quad \omega = \frac{v}{b} \rightarrow \Delta\theta = \frac{v\Delta t}{b}$$



so, $b\omega^2 = b \cdot \frac{v^2}{b^2} = \frac{v^2}{b}$

$$\frac{\Delta\theta}{\Delta t} = \frac{v}{b} = \omega$$

"Centripetal Acceleration"
Emphasize.

Generalize This

$$\vec{r}(t) = r\hat{r}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt} = \underline{\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \ddot{r}\hat{r} + \dot{r}\underbrace{\frac{d\hat{r}}{dt}}_{\dot{\theta}\hat{\theta}} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\underbrace{\frac{d\hat{\theta}}{dt}}_{-\hat{r}\dot{\theta}^2}$$

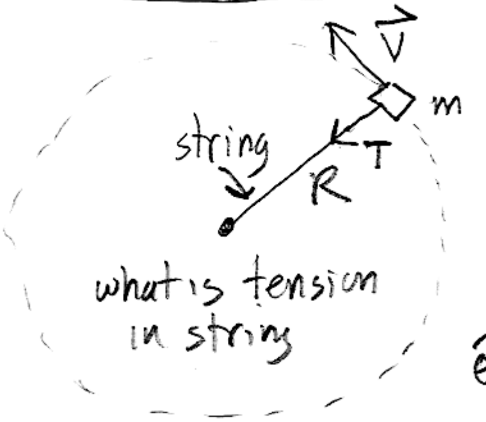
$$\boxed{\vec{a} = \overbrace{(\ddot{r} - r\dot{\theta}^2)}^{a_r}\hat{r} + \overbrace{(r\ddot{\theta} + 2\dot{r}\dot{\theta})}^{a_\theta}\hat{\theta}}$$

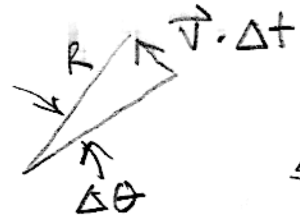
radial   centripetal   angular   "Coriolis"
accelerations   accelerations

Life is complicated in rotating coordinate systems!

Centripetal .. "seek center"
centrifugal .. "flee center"

Example 2.5 · Block on a string - focus on external view
· In outer space - no gravity





$$\Delta\theta = \frac{|\vec{V}|\Delta t}{R}$$

$$\frac{\Delta\theta}{\Delta t} = \dot{\theta} = \omega = \frac{|\vec{V}|}{R} = \frac{V}{R}$$

resolve along $\hat{r}$, $\hat{\theta}$

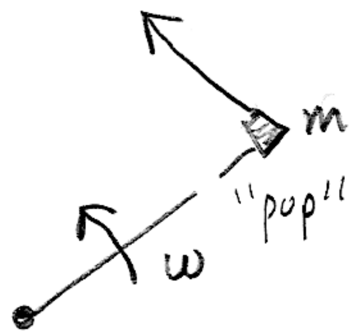
$$-T = m a_r = m(\cancel{\ddot{r}} - r\dot{\theta}^2)$$

$$-T = -mr\dot{\theta}^2 = -mR\cdot\frac{V^2}{R^2}$$

$$\boxed{T = m\frac{V^2}{R}}$$

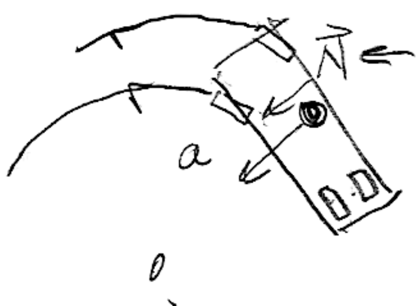
$$a_\theta = \cancel{r\ddot{\theta}} + \cancel{2\dot{r}\dot{\theta}}$$

Imagine: cut the string, mass flies out:



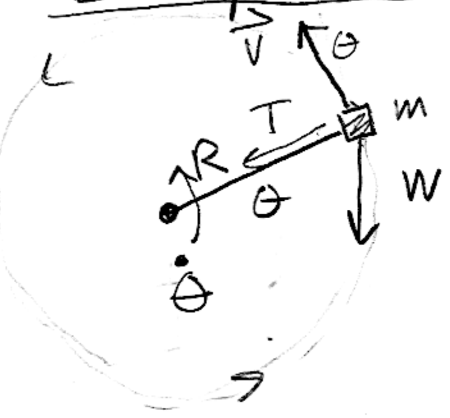
then

The tension keeps turning the mass about the origin
Going around a curve in a car



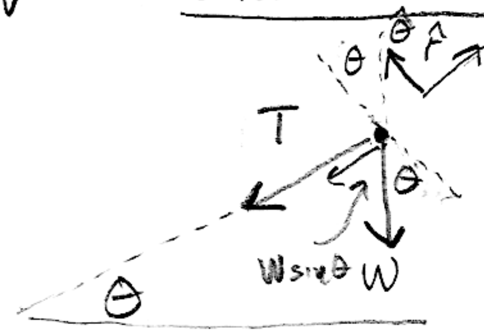
← normal force of door on you pushes you toward center, you interpret as being "thrown out"

Example 2.6 : Twirl on earth



what is tension now; it depends on θ.

Draw forces only:



along $\hat{r}$: $-T - W\sin\theta = ma_r = m(\ddot{r} - r\dot\theta^2)$

$-T - W\sin\theta = -mr\dot\theta^2$ $\qquad \dot\theta = \frac{v(t)}{R}$

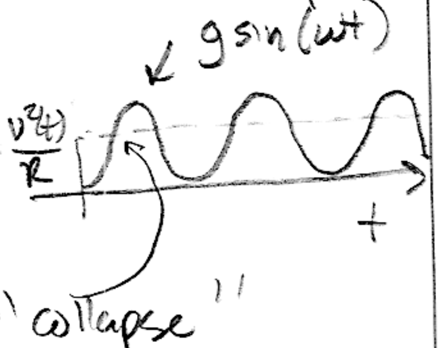
$T = mR\dot\theta^2 - mg\sin\theta$

$$\boxed{T = m\left[\frac{v^2(t)}{R} - g\sin\theta\right]}$$

tension component must be positive.. strings don't push

$\frac{v^2(t)}{R} - g\sin\theta > 0$

$v(t) > \sqrt{gR\sin\theta(t)}$



along $\hat\theta$ $\quad -W\cos\theta = r\ddot\theta + 2\dot{r}\dot\theta$

$$\boxed{-mg\cos\theta = R\frac{d^2\theta}{dt^2}}$$

differential equation to solve for θ... gives $\theta(t)$.