Power Series or Taylor Series

Kleppner p. 41-45

Motivation

$$(1+x)^2 = (1+x)(1+x) = 1 + 2x + x^2$$

when $x \ll 1$, $x^2 \ll x$

$$(1+x)^2 \simeq 1 + 2x$$ ← "power series approximation for $(1+x)^2$"

note: $(1+x)^0 = 1$

$(1+x)^1 = 1 + x = 1 + 1 \cdot x$

$(1+x)(1+x) = 1 + x + x + x^2 = 1 + 2x + x^2$

Pascal's Triangle

$$\begin{array}{cc}
0 & 1 \\
1 & 1 \quad 1 \\
2 & 1 \quad 2 \quad 1 \\
3 & 1 \quad 3 \quad 3 \quad 1 \\
4 & 1 \quad 4 \quad 6 \quad 4 \quad 1 \\
 & 1 \quad 5 \quad 10
\end{array}$$

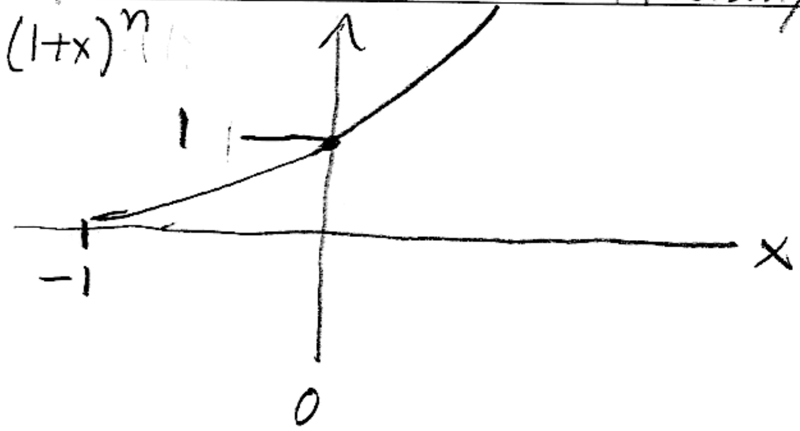
n ↗ ↖ n^{th} row: $0 + 1 + 2 + 3 + \dots n-1 = \frac{n(n-1)}{2} = \frac{1}{2}n(n-1)$

$$(1+x)^3 = (1+x)(1+2x+x^2) = 1 + 3x + 3x^2 + x^3$$

$$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + 1 \dots$$

$$(1+x)^n \simeq 1 + nx + \tfrac{1}{2}n(n-1) \cdot x^2 + \dots$$

$f(x) \to (1+x)^n$



Two views of approximation.

1. as done... $(1+x)^n \simeq 1 + nx + \frac{1}{2}n(n-1)x^2 \ldots$

2. $f(x) = (1+x)^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$

Choose a_i such that.

(i) $f(0) = 1 = a_0 + a_1 \cdot 0 + a_2 \cdot 0^2,\ \boxed{a_0 = 1 = f(0)}$

(ii) $f'(0) = n(1+\underbrace{x}_{0})^{n-1} = n = a_1 + 2a_2 \underbrace{x}_{0} + 3a_3 \underbrace{x^2}_{0} + \ldots$

$$\boxed{a_1 = n = f'(0)}$$

(iii) $f''(0) = n(n-1)(1+\underbrace{x}_{0})^{n-2} = n(n-1) = 2a_2 + 3 \cdot 2 a_3 \overbrace{x}^{0} + 4 \cdot 3 \cdot a_4 \overbrace{x^2}^{0} + \ldots$

$$\boxed{a_2 = \frac{1}{2} n(n-1) = \frac{1}{2} f''(0)}$$

(iv) $f'''(0) = n(n-1)(n-2) = 3 \cdot 2 \cdot 1 \cdot a_3$

$$\boxed{a_3 = \frac{1}{3 \cdot 2} n(n-1)(n-2) = \frac{1}{3 \cdot 2} \cdot f'''(0)}$$

Generalize 2 ways

any $f(x)$ with nice derivatives

$$f(x) = f(0) + f'(0)\cdot x + \frac{1}{2}f''(0)x^2 + \frac{1}{3\cdot 2}f'''(0)x^3 + \dots$$

$$(1+x)^{1/3} \simeq 1 + \frac{1}{3}\cdot x + \frac{1}{2}\cdot\frac{1}{3}\cdot\left(-\frac{2}{3}\right)x^2 + \frac{1}{2\cdot 3}\cdot\frac{1}{3}\cdot\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)x^3$$

n need not be integer

$$\simeq 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 + \dots$$

(mathematica)

What was so special about $x=0$

$x = x - 0$ = distance from 0

"about" point a, rather than zero,

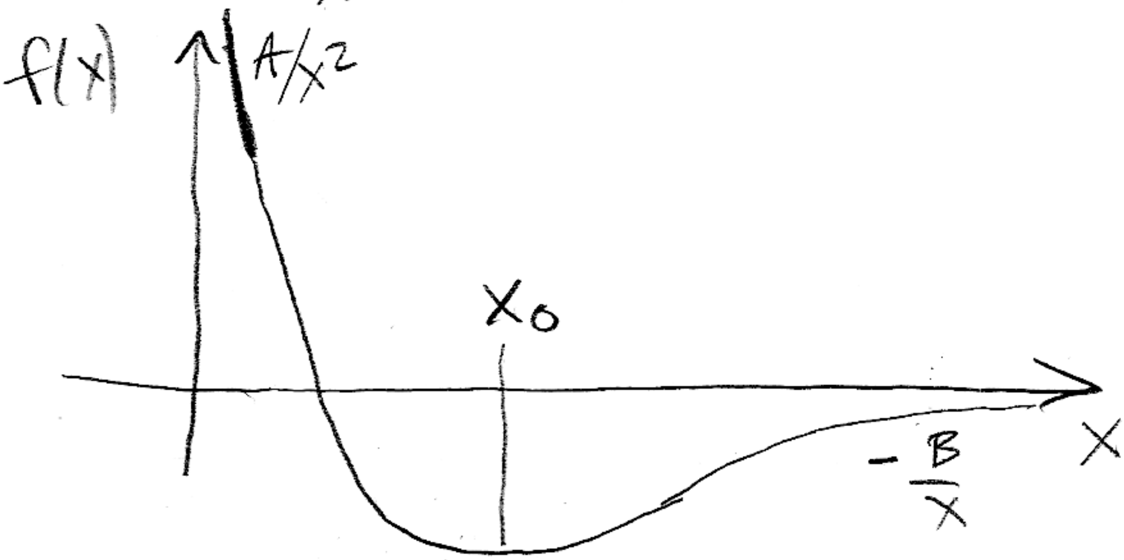
replace 0 with a

$x-0$ with $x-a$

$$f(x) = f(a) + f'(a)\cdot(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{2\cdot 3}\cdot f'''(a)(x-a)^3 + \dots$$

Minima (or Maxima) Special

$$f(x) = \frac{A}{x^2} - \frac{B}{x} \qquad \text{(orbital mechanics)}$$



↑ minimum in between.

$$f'(x) = \frac{df}{dx} = \frac{-2A}{x^3} + \frac{B}{x^2}, \quad \frac{-2A}{x_0^3} + \frac{B}{x_0^2} = 0$$

$$\frac{B}{x_0^2} = \frac{2A}{x_0^3}$$

$$\boxed{x_0 = 2\frac{A}{B}} \qquad A\uparrow,\ x_0\uparrow \quad B\uparrow,\ x_0\downarrow$$

$$f(x) \cong \frac{A}{x_0^2} - \frac{B}{x_0} + \underbrace{f'(x_0)}_{0} \cdot (x - x_0) + \ldots$$

$$\cong \frac{A}{4\frac{A^2}{B^2}} - \frac{B}{2\frac{A}{B}} + \frac{1}{2} f''(x_0)(x - x_0)^2$$

$$f(x) \cong \frac{1}{4}\frac{B^2}{A} - \frac{1}{2}\frac{B^2}{A} + \frac{1}{2}f''(x_0)(x-x_0)^2$$

$$f(x) \cong -\frac{1}{4}\frac{B^2}{A} + \frac{1}{2}f''(x_0)(x-x_0)^2$$

$$f''(x) = \frac{d^2f}{dx^2} = +\frac{6A}{x^4} - \frac{2B}{x^3}$$

$$f''(x_0) = \frac{6A}{16\frac{A^4}{B^4}} - \frac{2B}{8\frac{A^3}{B^3}} = \left(\frac{3}{8} - \frac{2}{8}\right)\frac{B^4}{A^3}$$

$$\boxed{f''(x_0) = \frac{1}{8}\frac{B^4}{A^3}}$$

$$f(x) \simeq -\frac{1}{4}\frac{B^2}{A} + \frac{1}{2}\cdot\frac{1}{8}\cdot\frac{B^4}{A^3}(x-x_0)^2$$

$$\simeq -\frac{1}{4}\frac{B^2}{A} + \frac{1}{2}\cdot\frac{1}{8}\cdot\frac{B^2}{A}\cdot\frac{B^2}{A^2}(x-x_0)^2$$

but $\frac{4A^2}{B^2} = x_0^2$

$$f(x) = -\frac{1}{4}\frac{B^2}{A} + \frac{1}{4}\frac{B^2}{A}\cdot\frac{(x-x_0)^2}{x_0^2}$$

$$\boxed{f(x) = \frac{1}{4}\frac{B^2}{A}\cdot\left(-1 + \frac{(x-x_0)^2}{x_0^2}\right)}$$