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Phys 20: Discussion 3
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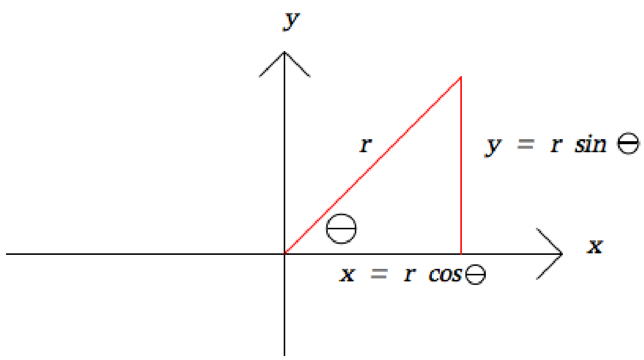
Polar Coordinates

I. Basis Vectors



A general vector in the 2D plane is

$$\vec{x} = x\hat{x} + y\hat{y} = (r\cos\theta)\hat{x} + (r\sin\theta)\hat{y} = r\underbrace{(\cos\theta\,\hat{x} + \sin\theta\,\hat{y})}_{\equiv\,\hat{r}}\ . \qquad (a)$$

We make the last definition from writing any vector $\vec{c}$ as $\vec{c} = c\,\hat{c}$, where $c \equiv |\vec{c}\,|$ is the magnitude of $\vec{c}$, and $\hat{c} \equiv \vec{c}/|\vec{c}\,|$ is the unit vector in the direction of $\vec{c}$.[1]

If $\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$, what is $\hat{\theta}$? Just as $\hat{r}$ is a linear combination of the two basis vectors $\hat{x}$ and $\hat{y}$, we must have $\hat{\theta}$ be a linear combination of $\hat{x}$ and $\hat{y}$. That is, $\hat{\theta} = \alpha\,\hat{x} + \beta\,\hat{y}$, where α and β are functions to be determined.

We want $\hat{\theta}$ to be perpendicular to $\hat{r}$, so we impose $\hat{\theta}\cdot\hat{r} = 0$. We also want $\hat{\theta}$ to be a unit vector, so we impose $\hat{\theta}\cdot\hat{\theta} = 1$. Therefore:

$$\hat{\theta}\cdot\hat{r} = (\alpha\,\hat{x} + \beta\,\hat{y})\cdot(\cos\theta\,\hat{x} + \sin\theta\,\hat{y}) = \alpha\cos\theta + \beta\sin\theta = 0 \qquad (b)$$

$$\hat{\theta}\cdot\hat{\theta} = (\alpha\,\hat{x} + \beta\,\hat{y})\cdot(\alpha\,\hat{x} + \beta\,\hat{y}) = \alpha^2 + \beta^2 = 1 \qquad (c)$$

In the above we have used $\hat{x}\cdot\hat{x} = 1$, $\hat{x}\cdot\hat{y} = \hat{y}\cdot\hat{x} = 0$ and $\hat{y}\cdot\hat{y} = 1$. From now on let $c \equiv \cos\theta$ and $s \equiv \sin\theta$ for brevity. Condition (b) implies $\alpha = -\beta s/c$, and putting that into condition (c) implies $(-\beta s/c)^2 + \beta^2 = 1 \implies \beta^2(s^2 + c^2) = c^2 \implies \beta^2 = c$ (since $s^2 + c^2 = 1$), or in other words $\beta = \pm c$. Putting this back into condition (b) gives $\alpha = \mp s$. So we have

$$\hat{\theta} = \pm(-\sin\theta\,\hat{x} + \cos\theta\,\hat{y}) \qquad (d)$$

[1] We use the standard (confusing) notation in which $\vec{x}$ is the general coordinate vector with magnitude $r \equiv |\vec{x}\,| = \sqrt{x^2 + y^2}$, where x is the first component of $\vec{x}$ and y is the second component of $\vec{x}$.

where the important aspect of the above equation so far is the relative sign difference between the $\hat{x}$ term and the $\hat{y}$ term. Which is the correct overall sign?

Consider a coordinate vector purely along the $\hat{x}$ direction, or in other words a coordinate vector for which $\theta = 0$.

We typically choose the convention for which "counterclockwise" is the positive angular direction, meaning that we want the above vector to be in the $+\hat{\theta}$ direction. Looking back at equation (d), we see that plugging in $\theta = 0$ gives

$$\hat{\theta}|_{\theta=0} = \pm(-0\,\hat{x} + 1\,\hat{y}) = \pm\hat{y}$$

so we should choose the overall plus sign. Therefore we have

$$\hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y} \qquad (e)$$

for the unit vector in the angular direction.

II. Time Derivatives

Summarizing equations (a) and (e), the unit vectors in 2D polar coordinates are

$$\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y} \qquad (f.1)$$
$$\hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}\ . \qquad (f.2)$$

What should strike you is that these unit vectors are functions of θ – in other words, these basis vectors are *not constant in space*. You can see this by just drawing unit vectors at each point on, say, a circle:
(draw)

But $\hat{x}$ and $\hat{y}$ are constant, you can use equations $(f.1)$ and $(f.2)$ to compute the time derivatives of $\hat{r}$ and $\hat{\theta}$. Let $\cdot \equiv \frac{d}{dt}$, where we use either notation whenever we feel like it. We find

$$\frac{d}{dt}\hat{r} = \frac{d\cos\theta}{dt}\,\hat{x} + \frac{d\sin\theta}{dt}\,\hat{y} = -\sin\theta\,\dot{\theta}\,\hat{x} + \cos\theta\,\dot{\theta}\,\hat{y} = \dot{\theta}(-s\,\hat{x} + c\,\hat{y}) = \dot{\theta}\,\hat{\theta} \qquad (g.1)$$
$$\frac{d}{dt}\hat{\theta} = -\frac{d\sin\theta}{dt}\,\hat{x} + \frac{d\cos\theta}{dt}\,\hat{y} = -\cos\theta\,\dot{\theta}\,\hat{x} - \sin\theta\,\dot{\theta}\,\hat{y} = -\dot{\theta}(c\hat{x} + s\hat{y}) = -\dot{\theta}\,\hat{r}\ . \qquad (g.2)$$

Given a general coordinate vector $\vec{x} = r\hat{r}$, we are now ready to compute the velocity vector $\vec{v} \equiv \dot{\vec{x}}$ and the acceleration vector $\vec{a} \equiv \dot{\vec{v}} = \ddot{\vec{x}}$ in polar coordinates. First the velocity vector:

$$\vec{v} = \dot{\vec{x}} = \frac{d}{dt}(r\hat{r}) = \dot{r}\,\hat{r} + r\frac{d}{dt}\hat{r} = \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta} \equiv v_r\,\hat{r} + v_\theta\,\hat{\theta}$$
$$\Longrightarrow\ v_r = \dot{r}\ ,\quad v_\theta = r\dot{\theta} \qquad (h)$$

Then the acceleration vector:

$$\vec{a} = \dot{\vec{v}} = \frac{d}{dt}(\dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta}) = \ddot{r}\,\hat{r} + \dot{r}\frac{d}{dt}\hat{r} + \dot{r}\dot{\theta}\,\hat{\theta} + r\ddot{\theta}\,\hat{\theta} + r\dot{\theta}\frac{d}{dt}\hat{\theta}$$
$$= \ddot{r}\,\hat{r} + \dot{r}(\dot{\theta}\,\hat{\theta}) + \dot{r}\dot{\theta}\,\hat{\theta} + r\ddot{\theta}\,\hat{\theta} + r\dot{\theta}(-\dot{\theta}\,\hat{r}) = (\ddot{r} - r\dot{\theta}^2)\,\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{\theta}$$

$$\implies a_r = \ddot{r} - r\dot{\theta}^2\ , \quad a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta} \qquad (i)$$

Equation (i) is the point of this lecture. In addition to the "obvious" terms $\ddot{r}$ and $r\ddot{\theta}$, we have the additional terms $-r\dot{\theta}^2$ and $2\dot{r}\dot{\theta}$, which exist purely because the basis vectors $\hat{r}$ and $\hat{\theta}$ are not constant in space.

III. Example Trajectories

Suppose the trajectory of a ball is such that $\ddot{r} = \ddot{\theta} = 0$. What would the path of the ball look like if...

A) $\dot{r} \neq 0$ but $\dot{\theta} = 0$?

B) $\dot{r} = 0$ but $\dot{\theta} \neq 0$?

C) both $\dot{r} \neq 0$ and $\dot{\theta} \neq 0$?

For the first one, the ball is moving radially outward at a fixed angle. Nothing particularly interesting there.

For the second one, the condition $\dot{r} = 0$ means that the ball is not moving in the radial direction. In other words, it is always at a fixed distance from the origin. However, the time rate of change of its angular position is some nonzero value (assume that it is constant for simplicity). The ball is moving in a circle!

For the third one, the condition $\dot{r} \neq 0$ means that the ball is moving in the radial direction. If it starts at the origin ($r = 0$), for example, then the ball moves in a spiral outward from the origin:

```
(* x = r*Cos[θ], y = r*Sin[θ]  *)
(* if rdot = μ = const and θdot = ω = const,
then r = μ*t and θ = ω*t *)

μ = 1;
ω = 1;
ParametricPlot[{μ*t*Cos[ω*t], μ*t*Sin[ω*t]}, {t, 0, 10}]
```