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Phys 20: Discussion 2
9/30/2010

Cross Product

I. Invariant Tensors

We defined a rotation as something that leaves the "length" $v^2 \equiv \sum_{i=1}^{3} v_i^2$ invariant. Another way to say this is that a linear operator M is called a rotation matrix if it leaves the symbol δ_{ij} invariant. What condition must such an M satisfy? Well, take δ_{ij} and perform the operation M (one matrix for each index!):

$$\delta_{ij} \to M_{ik} M_{j\ell} \delta_{k\ell} = (MIM^T)_{ij}$$

If δ_{ij} is to be invariant, then we want $MIM^T = I$, or in other words $MM^T = I$. That is, a matrix M will leave δ_{ij} invariant if

$$M^T = M^{-1} .$$

Matrices that satisfy this condition are called "orthogonal."

Without motivation, let's stare at another object, this time the completely antisymmetric symbol ε_{ijk}, which I remind you is defined as equalling $+1$ for cyclic permutations of $ijk = 123$, equalling -1 for cyclic permutations of $ijk = 321$, and equalling zero if any of the two indices are equal. We thought of rotations as things that leave δ_{ij} invariant; what types of operations will leave ε_{ijk} invariant?

Suppose the transformation M leaves ε_{ijk} invariant. In math, these words mean:

$$\varepsilon_{ijk} \to M_{ia} M_{jb} M_{kc} \varepsilon_{abc} = \varepsilon_{ijk}$$

Fun fact of life: $\varepsilon_{ijk}\varepsilon_{ijk} = 3!$ (work it out numerically). So, take both sides of the above equation and multiply by ε_{ijk} (summing over i, j, k as usual) to get

$$\varepsilon_{ijk}\varepsilon_{abc} M_{ia} M_{jb} M_{kc} = 3!$$

In other words, the transformation M will leave the symbol ε_{ijk} unchanged provided that M satisfies

$$\frac{1}{3!}\varepsilon_{ijk}\varepsilon_{abc} M_{ia} M_{jb} M_{kc} = 1 .$$

This strange looking requirement is typically repackaged using the language "determinant." For any matrix M, its determinant is defined as

$$\det M \equiv \frac{1}{3!}\varepsilon_{ijk}\varepsilon_{abc} M_{ia} M_{jb} M_{kc} .$$

If you have seen determinants before, you should check that this definition corresponds in perhaps unfamiliar language to what you already know.

Terminology: **From now on, we will reserve the word "rotation" for any operation M that leaves invariant both the identity tensor δ_{ij} and the antisymmetric tensor ε_{ijk}.**

We are including the ε_{ijk} invariance condition, or equivalently the condition $\det M = 1$, because rotations for which $\det M = -1$ instead of $+1$ are rotations composed with reflections, and I don't want to deal with reflections right now. In fancy talk, the set of matrices M that satisfy $M^T = M^{-1}$ and $\det M = 1$ is called the group $SO(3)$, where the S means "special" (meaning $\det M = 1$)," the O means "orthogonal" (meaning $M^T = M^{-1}$), and the 3 stands for the size of the matrix, namely 3-by-3.

Example: Compute the determinant of the 2-by-2 matrix

$$M = \begin{pmatrix} w & x \\ y & z \end{pmatrix} .$$

Answer:

$$\begin{aligned} \det M &= \frac{1}{2}\varepsilon_{ij}\varepsilon_{ab}M_{ia}M_{jb} = \frac{1}{2}\varepsilon_{ij}(M_{i1}M_{j2} - M_{i2}M_{j1}) \\ &= \frac{1}{2}\left[(M_{11}M_{22} - M_{21}M_{12}) - (M_{12}M_{21} - M_{22}M_{11})\right] \\ &= M_{11}M_{22} - M_{12}M_{21} = wz - xy \end{aligned}$$

II. Cross Product

I had a purpose for all that annoying formalism, I promise. Here it is. Let $\{v_i\}_{i=1}^3$ be the components of a vector $\vec{v}$, and let $\{w_i\}_{i=1}^3$ be components of a vector $\vec{w}$. Use the fancy epsilon tensor you learned about to construct the new set of numbers $c_i \equiv \varepsilon_{ijk}v_jw_k$. The thing on the left has one free tensor index and thus the three numbers $\{c_i\}_{i=1}^3$ are the components of a vector.

The vector $\vec{c}$ with components $c_i = \varepsilon_{ijk}v_jw_k$ is what is called "the cross product" of the two vectors $\vec{v}$ and $\vec{w}$.

Example:

Let $v_i = \begin{pmatrix} 5 \\ 4 \\ 1 \end{pmatrix}$, $w_i = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$. Compute $\vec{v} \times \vec{w}$.

Solution:

$$[\vec{v} \times \vec{w}]_1 = \varepsilon_{1jk} = \varepsilon_{123}v_2w_3 + \varepsilon_{132}v_3w_2 = (+1)v_2w_3 + (-1)v_3w_2 = 4 \times 3 - 1 \times 2 = 12 - 2 = 10$$
$$[\vec{v} \times \vec{w}]_2 = \varepsilon_{2jk}v_jw_k = v_3w_1 - v_1w_3 = 1 \times 0 - 5 \times 3 = 15$$
$$[\vec{v} \times \vec{w}]_3 = \varepsilon_{3jk}v_jw_k = v_1w_2 - v_2w_1 = 5 \times 2 - 4 \times 0 = 10$$

Therefore

$$(\vec{v} \times \vec{w})_i = \begin{pmatrix} 10 \\ -15 \\ 10 \end{pmatrix} .$$

Once you get enough practice playing with that epsilon tensor, you'll be able to compute cross products in seconds.

What else can we do? Remember that ε_{ijk} is invariant under rotations. We started off with using the invariant symbol δ_{ij} to define the "length" of a vector $v^2 \equiv \delta_{ij} v_i v_j$, or more generally the scalar product between two vectors, $\vec{v} \cdot \vec{w} \equiv \delta_{ij} v_i w_j = v_1 w_1 + v_2 w_2 + v_3 w_3$, which is also invariant under rotations. Take three vectors $\vec{a}, \vec{b}$ and $\vec{c}$. Three vectors, eh? Well ε_{ijk} has vector indices (that is, it transforms like the product of three vectors), and it is invariant under rotations... so let's follow our nose and write down

$$V \equiv \varepsilon_{ijk} a_i b_j c_k .$$

What is that? Before spoiling it for you, notice that based on our definition of the cross product we can repackage the above as

$$V = \vec{a} \cdot (\vec{b} \times \vec{c}) .$$

That thing is the volume of the parallelepiped enclosed by the three vectors $\vec{a}, \vec{b}$ and $\vec{c}$. Nice, right?

III. Test your Understanding

1. Derive the seemingly mysterious identity

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{v}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) - \nabla^2 \vec{v}$$

for any vector $\vec{v}$. ($\vec{\nabla} \equiv \hat{e}_i \frac{\partial}{\partial x_i}$ and $\nabla^2 \equiv \vec{\nabla} \cdot \vec{\nabla}$, in case you haven't seen those symbols.)

Hint: First prove to yourself that $\varepsilon_{ijk} \varepsilon_{kab} = \delta_{ia} \delta_{jb} - \delta_{ib} \delta_{ja}$.

2. Look very carefully at what we've done in discussions 1 and 2 for any mention of the fact that we are considering 3 dimensions. Sometimes it mattered, and sometimes it did not. Where did it matter? For example, try to generalize the cross product to 4 dimensions. What do you get?

3. Go through everything we did so far in 3 dimensions, except everywhere you see δ_{ij} replace it with

$$\eta_{ij} \equiv \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

What changes? What doesn't?