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Notation, Rotations, Matrices

0. Notation

Everything will be in 3 dimensions, and in Cartesian coordinates (x, y, z).

"$\equiv$" means "defined as" or "equivalent to," depending on context.

$x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv z$; $\hat{e}_1 \equiv \hat{x}$, $\hat{e}_2 \equiv \hat{y}$, $\hat{e}_3 \equiv \hat{z}$

$$\vec{x} \equiv x\hat{x} + y\hat{y} + z\hat{z} \equiv x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3 \equiv \sum_{j=1}^{3} x_j\hat{e}_j \equiv x_i\hat{e}_j$$

$$r^2 \equiv |\vec{x}\,|^2 \equiv \vec{x}\cdot\vec{x} \equiv x_1^2 + x_2^2 + x_3^2 \equiv \sum_{i=1}^{3} x_i x_i = \sum_{i=1}^{3}\sum_{j=1}^{3} \delta_{ij} x_i x_j \equiv \delta_{ij} x_i x_j$$

$$= \vec{x}^T I \vec{x} = (x_1, x_2, x_3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\delta_{ij} \equiv \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \leftarrow \text{“Kronecker delta”}$$

$$\varepsilon_{ijk} \equiv \begin{cases} +1 \text{ if } ijk = 123,\ 312,\ 231 \\ -1 \text{ if } ijk = 321,\ 132,\ 213 \\ 0 \text{ if any two indices are equal} \end{cases} \leftarrow \text{“Levi-Civita tensor” or “epsilon tensor”}$$

The subscripts ijk etc are called "indices."

M^T is called "the transpose of M," meaning interchange the rows and columns:$(M^T)_{ij} = M_{ji}$.

I. Rotations

A sphere or radius r is the set of all points $\{x_i\}_{i=1}^{3}$ for which $r^2 \equiv x_1^2 + x_2^2 + x_3^2$ is fixed.

A rotation is an operation that leaves the sphere invariant, or in other words takes you from one point on the sphere to another point on the sphere. To clarify: Take any point on the sphere: $\{x_i\}_{i=1}^{3}$. Operating by some linear transformation M acts as $x_i \to M_{ij}x_j$. For M to be called a rotation, we require $r^2 \to r^2$ under M. That is, $r^2 \equiv \delta_{ij}x_i x_j$ is unchanged.

Example 1:

A rotation through an angle θ about the z-axis is implemented by the matrix

$$R(\hat{z}, \theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Rotate the vector $\vec{x} = (1)\hat{e}_x + (0)\hat{e}_y + (0)\hat{e}_z$, or in other words $v_i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, through an angle θ about the z-axis. Show that $r^2 \equiv \delta_{ij} x_i x_j = \sum_{i=1}^{3} x_i^2$ is unchanged by the rotation.

Solution: The rotation acts as

$$x_i \to R_{ij} x_j = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} .$$

Before the rotation, $r^2 = \sum_{i=1}^{3} x_i^2 = x_1^2 + 0 + 0 = 1$. After the rotation, $r^2 = (\cos\theta)^2 + (\sin\theta)^2 + 0^2 = 1$. It works.

Example 2:

Do the same as in Example 1, but this time for the object $\vec{v} = a\hat{x} + b\hat{y} + c\hat{z}$, or in other words $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, where a, b, c are totally arbitrary real numbers.

Solution:

The rotation acts as

$$v_i \to R_{ij} v_j = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a\cos\theta - b\sin\theta \\ a\sin\theta + b\cos\theta \\ c \end{pmatrix} .$$

Before, $v^2 \equiv \sum_{i=1}^{3} v_i^2 = a^2 + b^2 + c^2$. After,

$$\begin{aligned} v^2 &\to (a\cos\theta - b\sin\theta)^2 + (a\sin\theta + b\cos\theta)^2 + c^2 \\ &= a^2(\cos\theta)^2 + b^2(\sin\theta)^2 - 2ab\,\cos\theta\sin\theta + a^2(\sin\theta)^2 + b^2(\cos\theta)^2 + 2ab\,\cos\theta\sin\theta + c^2 \\ &= a^2[(\sin\theta)^2 + (\cos\theta)^2] + b^2[(\sin\theta)^2 + (\cos\theta)^2] - 2ab\,\cos\theta\sin\theta + 2ab\,\sin\theta\cos\theta + c^2 \\ &= a^2 + b^2 + c^2 \ \checkmark \end{aligned}$$

Notice that in the first example we rotated a set of coordinates $\{x_i\}_{i=1}^3$, while in the second example we rotated some other set of 3 numbers $\{v_i\}_{i=1}^3$ expressed in terms of the same directions $\{\hat{e}_i\}_{i=1}^3$ as the coordinates $\{x_i\}_{i=1}^3$. We take this as our definition of a vector:

A vector is something whose components transform under rotations in exactly the same way as the coordinates do.

In other words, a vector is something whose 3 components have one index (subscript) that transforms under rotations, denoted by $\{v_i\}_{i=1}^3$. If we feel like it, we can also define a collection of $3 \times 3 = 9$ numbers that transform under rotations, call them $\{t_{ij}\}$ where i and j each run from 1 to 3 (just like the subscript on v_i). Such an object $t = t_{ij}\hat{e}_i\hat{e}_j$ is called a "tensor."

It's just a fancy word for the straightforward generalization of a vector. In fact, you have already done this with the symbol δ_{ij}. Who cares? You will, when you see the "moment of inertia tensor" assail your well-being.

Similarly, you can play this game with as many indices as you want: define $3 \times 3 \times 3 = 27$ numbers $\{\omega_{ijk}\}$ that transform appropriately under rotations. To specify the number of indices, you can if you want to sound cool choose to say the words "tensor of rank n" for a collection of objects carrying n indices, such as $K_{i_1 \ldots i_n}$, where each $i_1, \ldots, i_n$ runs from 1 to 3. But you don't have to use those words. I usually don't bother.

III. Matrices

Matrix algebra is a set of rules for manipulating collections of numbers in such a way that turns out to be convenient in many applications. We have already used some matrix multiplication rules above, but here is a crash-course in how to manipulate these things if you are unfamiliar with them.

Rule: Vectors are packaged as columns. The components $\{v_i\}_{i=1}^3$ are written as

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} .$$

Sometimes this notation is confusing because the basis is never explicitly displayed. That is, a vector $\vec{v} = v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3$ is displayed as the above column, but nowhere in the notation are the basis vectors $\{\hat{e}_i\}_{i=1}^3$ written.

Rule: Matrix multiplication is defined as "row times column." Example: Consider the identity matrix

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

(Rule: In a matrix such as δ_{ij}, the first index i labels the row, and the second index j labels the column. So, for example, δ_{12} means "row 1, column 2" which is zero.)

Take the vector $\vec{v}$ whose components are $(v_1, v_2, v_3)^T$ as above. (Incidentally, rule: a superscript T means you switch all rows and columns. So $(\text{row})^T$ means column. We wrote it that way for typographical convenience only.) Multiplying $\vec{v}$ by the identity matrix better just give $\vec{v}$ (definition of multiplicative identity: thing$\times I =$ thing). Row times column, row times column, row times column:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 \times v_1 + 0 \times v_2 + 0 \times v_3 \\ 0 \times v_1 + 1 \times v_2 + 0 \times v_3 \\ 0 \times v_1 + 0 \times v_2 + 1 \times v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \checkmark$$

Work out $\sum_{j=1}^3 \delta_{ij} v_j$ by expanding out the sum over j, then plug in $i = 1$, then $i = 2$, then $i = 3$. You should find that in doing so you are performing exactly the same sequence of steps as when doing the above matrix multiplication. The two notations are different ways to write identical arithmetic operations. Another example:

$$\begin{pmatrix} a & b & c \\ 1 & 2 & 3 \\ x & y & z \end{pmatrix} \begin{pmatrix} \psi \\ \chi \\ \varphi \end{pmatrix} = \begin{pmatrix} a\psi + b\chi + c\varphi \\ \psi + 2\chi + 3\varphi \\ x\psi + y\chi + z\varphi \end{pmatrix} .$$

It is useful to know how to input these things into Mathematica to check your work. The syntax for the above example is (human math language on the left, Mathematica syntax on the right):

$$\begin{pmatrix} a & b & c \\ 1 & 2 & 3 \\ x & y & z \end{pmatrix} \leftrightarrow \{\{a, b, c\}, \{1, 2, 3\}, \{x, y, z\}\}$$

$$\begin{pmatrix} \psi \\ \chi \\ \varphi \end{pmatrix} \leftrightarrow \{: psi :, : chi :, : phi :\}$$

Note that Mathematica does not distinguish between column vectors and row vectors; it uses its magic to figure out what to do. You, on the other hand, do not have magic, so keep careful track of rows and columns. The syntax for multiplying a matrix M and a matrix N in Mathematica is "M.N" (See the dot down there? You need it. Otherwise it will just multiply entry by entry... why? I don't know). There is plenty more to say, but let's leave it at that for now.