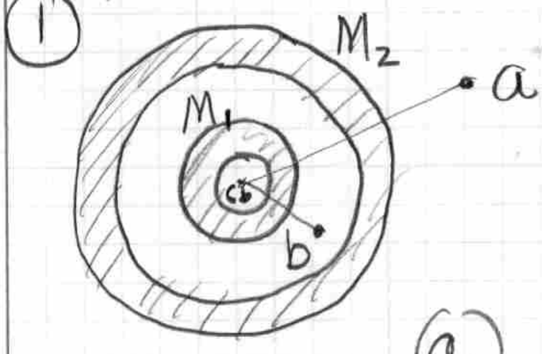
(1)



All force directions toward center, or, "$-\hat{r}$"

(a) $\vec{F} = -\dfrac{G(M_1 + M_2)m}{a^2}\hat{r}$

since shells uniform and all mass at radii < a, as if all mass concentrated at origin.

(b) $\vec{F} = -\dfrac{G M_1 m}{b^2}\hat{r}$

(c) $\vec{F} = 0$ no mass inside

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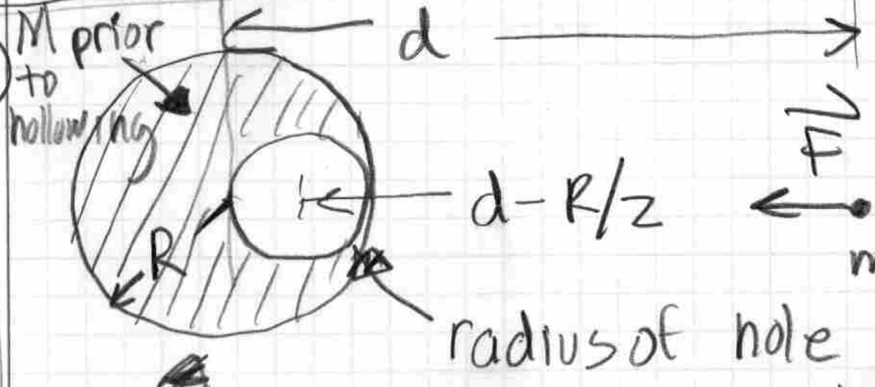
(3)

(2) after (4)



radius of hole is − R/2

∴ volume is $\frac{1}{8}$ of big sphere's volume. ... $(\frac{1}{2})^3$

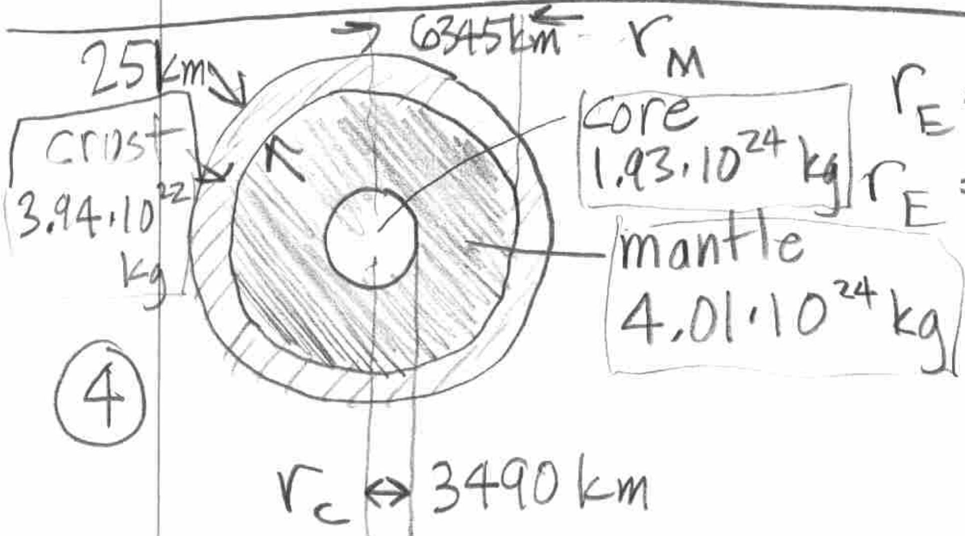
use concept of superposition ...

"LIKE" full sphere of radius R + − mass sphere of 1/8 mass at R/2

$$\vec{F} = \left(-\frac{GMm}{d^2} + \frac{G\left(\frac{M}{8}\right)m}{(d - R/2)^2}\right)\hat{r}$$

$\uparrow$ unit away from center

$$\vec{F} = -\frac{GMm}{d^2}\left[1 - \frac{1}{8\left(1 - \frac{R}{2d}\right)^2}\right]\hat{r}$$



$r_E = r_M + 25$ km

$r_E = 6370$ km

(4)

(a) $M_{tot} = M_{crust} + M_{mantle} + M_{core}$

$= (0.0394 + 4.01 + 1.93) \cdot 10^{24}$ kg

$= 5.98 \cdot 10^{24}$ kg

$$g = \frac{GM_{tot}}{R_E^2} = \frac{(6.673 \cdot 10^{-11})(5.98 \cdot 10^{24})}{(6370 \cdot 10^3)^2}$$

$$\boxed{g = 9.83 \text{ m/s}^2}$$

(b) $M_{inside} = M_{mantle} + M_{core}$

$= (4.01 + 1.93) \cdot 10^{24}$ kg

$= 5.94 \cdot 10^{24}$ kg

$$R = r_m = 6345 \text{ km}$$

$$g = \frac{G M_{inside}}{r_m^2} = \frac{(6.673 \cdot 10^{-11})(5.94 \cdot 10^{24})}{(6345 \cdot 10^3)^2}$$

$$\boxed{g = 9.85 \text{ m/s}^2}$$

(c) $$g = g_s\left(1 - \frac{D}{R_E}\right)$$

$$= 9.83\left(1 - \frac{25}{6370}\right)$$

$$\boxed{g = 9.79 \text{ m/s}^2}$$

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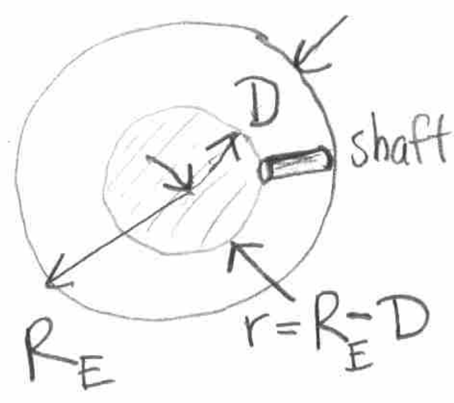
2 $$\rho_E = \frac{M_E}{V_E} = \frac{M_E}{\frac{4\pi}{3} R_E^3}$$



Mass inside D is:

$$M_{inside} = \frac{4\pi}{3} \rho_E (R_E - D)^3$$

$$g = \frac{G M_{inside}}{r^2}$$

$$= \frac{4\pi}{3} G \rho_E \frac{(R_E - D)^3}{(R_E - D)^2}$$

$$= \frac{\cancel{4\pi}}{\cancel{3}} G \frac{M_E}{\frac{\cancel{4\pi}}{\cancel{3}} R_E^3} (R_E - D) = \underbrace{\frac{G M_E}{R_E^2}}_{g_s}\left(1 - \frac{D}{R_E}\right)$$

$$\boxed{g = g_s\left(1 - \frac{D}{R_E}\right)}$$

5. a

Boundary	$\tilde{r}$ (boundary, km)	$\rho_<$ $(10^3 \frac{kg}{m^3})$	$\rho_>$ $(10^3 \frac{kg}{m^3})$
Core/Mantle	3490	10.8	4.50
Mantle/Crust	6345	5.55	3.10
Crust/Air	6375	5.52	$\simeq 0$

This is $\dfrac{\sum_i M_i \text{(inside)}}{\frac{4\pi}{3} r^3_{\text{boundary}}}$

This is $\dfrac{M_i \text{(just outside)}}{\frac{4\pi}{3}\left(r^3_{\text{next boundary}} - r^3_{\text{boundary}}\right)}$

b The mass inside radius r has two contributions, consisting of

i mass inside $\tilde{r}$ $= \frac{4\pi}{3}\rho_< \tilde{r}^3$

ii mass between $\tilde{r}$ and r:

$$= \frac{4\pi}{3}\rho_>\left(r^3 - \tilde{r}^3\right)$$

$$g = \frac{G}{r^2} M_{\text{inside}} = \frac{G}{r^2}\left(\frac{4\pi}{3}\rho_< \tilde{r}^3 + \frac{4\pi}{3}\rho_>\left(r^3 - \tilde{r}^3\right)\right)$$

$$g = G\left[\frac{4\pi}{3}\frac{(\rho_< - \rho_>)\tilde{r}^3}{r^2} + \frac{4\pi}{3}\rho_> r\right]$$

so $\boxed{A = \frac{4\pi}{3}(\rho_< - \rho_>)\tilde{r}^3 \qquad B = \frac{4\pi}{3}\rho_>}$

but we know from problem 4 that when $\rho_< = \rho_> = \rho$ $\quad g = \frac{4\pi}{3}G\rho r$, so, when ρ uniform, A must be zero, and it is.

(c) $\quad g = \frac{G M_{inside}}{r^2}$

$M_{inside} = 1.93 \cdot 10^{24}$ kg (core)

$r = 3490$ km (core)

$$g_c = \frac{(6.673 \cdot 10^{-11})\, 1.93 \cdot 10^{24}}{(3490 \cdot 10^3)^2}$$

$$\boxed{g_c = 10.6\ m/s^2}$$

at core/mantle boundary

from $r = 0$ to $r = r_c$ (core mantle boundary)

$$M_{inside} = \rho_{core} \cdot \frac{4\pi}{3} r^3 = \frac{M_{core}}{\frac{4\pi}{3} r_c^3} \cdot \frac{4\pi}{3} r^3 = M_{core}\left(\frac{r}{r_c}\right)^3$$

$$g(r) = \frac{G M_{inside}}{r^2} = \frac{G M_{core}}{r^2}\left(\frac{r}{r_c}\right)^3 = \frac{G M_{core}}{r_c^2}\left(\frac{r}{r_c}\right)$$

$$\boxed{g(r) = g_c\left(\frac{r}{r_c}\right) = (10.6\ m/s^2)\left(\frac{r}{3490\ km}\right)}$$

(d) $$g = G\left(\frac{A}{r^2} + Br\right)$$

$$\frac{dg}{dr} = G\left(-\frac{2A}{r_{min}^3} + B\right) = 0$$

$$\boxed{r_{min} = \left(\frac{2A}{B}\right)^{1/3} = \left(2\left[\frac{\rho_<}{\rho_>} - 1\right]\right)^{1/3}\tilde{r}}$$

$$g_{min} = G\left(\frac{A}{\left(\frac{2A}{B}\right)^{2/3}} + B\left(\frac{2A}{B}\right)^{1/3}\right)$$

$$= G\left(\frac{1}{2^{2/3}}(B^2A)^{1/3} + 2^{1/3}(B^2A)^{1/3}\right)$$

$$\boxed{g_{min} = G\left(\frac{1}{2^{2/3}} + 2^{1/3}\right)(AB^2)^{1/3}}$$

$$= \frac{4\pi}{3}G\left(\frac{1}{2^{2/3}} + 2^{1/3}\right)\left((\rho_< - \rho_>)\rho_>^2\tilde{r}^3\right)^{1/3}$$

$$\boxed{g_{min} = \frac{4\pi}{3}G\rho_>\tilde{r}\left(\frac{1}{2^{2/3}} + 2^{1/3}\right)\left(\frac{\rho_<}{\rho_>} - 1\right)^{1/3}}$$

$$\frac{d^2g}{dr^2} = G\,\frac{6A}{r_{min}^4} > 0,$$ above is a minimum;

need $\frac{\rho_<}{\rho_>} > 1$,

higher density inside.

for $r_{min} > \tilde{r}$, need

$$\left(2\left[\frac{\rho_<}{\rho_>} - 1\right]\right)^{1/3} > 1 \Rightarrow \boxed{\frac{\rho_<}{\rho_>} > \frac{3}{2}}$$

numerically,

$$r_{min} = \left(2\cdot\left(\frac{10.8}{4.50}-1\right)\right)^{1/3}\cdot 3490\ \text{km}$$

$$\boxed{r_{min} = 4932\ \text{km}}$$

$$g_{min} = \frac{4\pi}{3}\cdot(6.673\cdot10^{-11})(4.50\cdot10^{3})(3490\cdot10^{3})$$

$$\times\left(\frac{1}{2^{2/3}}+2^{1/3}\right)\left(\frac{10.8}{4.50}-1\right)^{1/3}$$

$$\boxed{g_{min} = 9.30\ \text{m/s}^2}$$

(e) core: $g(r) = \left(10.6\frac{m}{s^2}\right)\left(r/3490\ \text{km}\right)$

mantle: $g(r) = \underbrace{\frac{4\pi}{3}G(\rho_<-\rho_>)\tilde{r}}\cdot\left(\frac{\tilde{r}}{r}\right)^2 + \underbrace{\frac{4\pi}{3}G\rho_>\tilde{r}}\cdot\left(\frac{r}{\tilde{r}}\right)$

$$= \frac{4\pi}{3}\cdot 6.673\cdot10^{-11}(10.8-4.5)\cdot10^{3}\cdot(3490\cdot10^{3})$$

$$= 6.19\ \text{m/s}^2$$

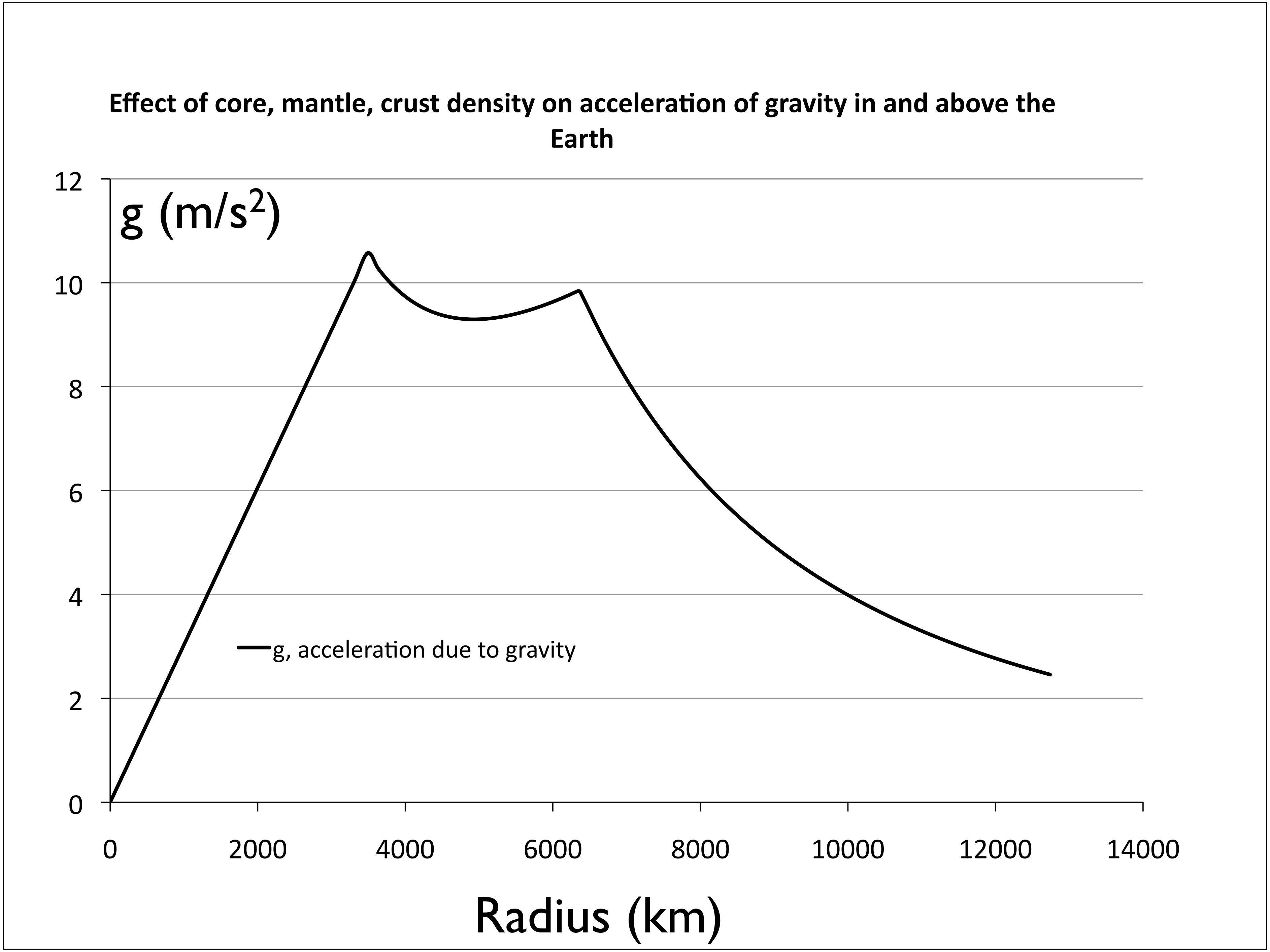
$$= \frac{4\pi}{3}(6.673\cdot10^{-11})(4.5\cdot10^{3})\cdot(3490\cdot10^{3})$$

$$= 4.39\ \text{m/s}^2$$

$$g(r) = \left(6.19\frac{m}{s^2}\right)\left(\frac{3490\ \text{km}}{r}\right)^2 + (4.39\ \text{m/s}^2)\left(\frac{r}{3490\ \text{km}}\right)$$

crust: $\frac{4\pi}{3}G(\rho_<-\rho_>)\tilde{r} = \frac{4\pi}{3}(6.673\cdot10^{-11})(5.55-3.10)\cdot10^{3}(6345\cdot10^{3})$

$$= 4.34\ \text{m/s}^2$$


Effect of core, mantle, crust density on acceleration of gravity in and above the Earth
g (m/s2)
12
10
8
6
4
2
0
g, acceleration due to gravity
0
2000
4000
6000
8000
10000
12000
14000
Radius (km)

$$\frac{4\pi}{3} G \rho_{>} \tilde{r} = \frac{4\pi}{3}(6.673\cdot 10^{-11})(3.10\cdot 10^{3})\cdot(6345\cdot 10^{3})$$

$$= 5.50 \ \mathrm{m/s^2}$$

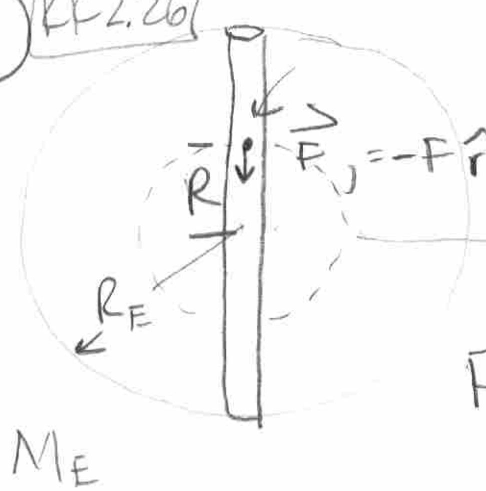
$$g(r) = \left(4.34\frac{\mathrm{m}}{\mathrm{s^2}}\right)\left(\frac{6345\ \mathrm{km}}{r}\right)^2 + \left(5.50\frac{\mathrm{m}}{\mathrm{s^2}}\right)\left(\frac{r}{6345\ \mathrm{km}}\right)$$

air:

$$g = \frac{GM_E}{r^2} = \underbrace{\frac{GM_E}{R_E^2}}_{g_s} \cdot \left(\frac{R_E}{r}\right)^2$$

$$\boxed{g(r) = \left(9.83\frac{\mathrm{m}}{\mathrm{s^2}}\right)\left(\frac{6370\ \mathrm{km}}{r}\right)^2}$$

6 KK2.26



$$\rho_E = \frac{M_E}{\frac{4\pi}{3} R_E^3}$$

$$M_{inside} = \frac{4\pi}{3}\rho_E R^3$$

$$F = G\frac{M_{inside}\, m}{R^2}$$

$$= G\frac{\frac{4\pi}{3}\rho_E R^3 \cdot m}{R^2}$$

$$= \underbrace{\left(G\frac{4\pi}{3}\rho_E R_E\right)}\left(\frac{R}{R_E}\right)\cdot m$$

$$= \underbrace{G\frac{M_E}{R_E^2}}_{g} \cdot \left(\frac{R}{R_E}\right)\cdot m$$

$$F = \left(\frac{R}{R_E}\right) mg = \underbrace{\left(\frac{mg}{R_E}\right)}_{\text{"}k\text{"}} R$$

direction is $-$,

$$\vec{F} = -kR\,\hat{r}$$

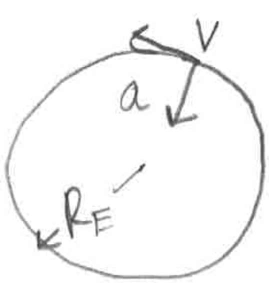
$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{mg}{mR_E}} = \sqrt{\frac{g}{R_E}}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R_E}{g}}$$

$$= 2\cdot\pi\cdot\sqrt{\frac{6370\cdot10^3}{9.83}}$$

$$\boxed{\begin{aligned} T &= 5060\ \text{s} = 84.3\,\text{min} \\ &= 1\ \text{hour}\ 24\ \text{minutes}\end{aligned}}$$

Low Earth Orbit



$$F_{Net} = ma$$

$$-\frac{GM_E \cancel{m}}{R_E^2} = -\cancel{m}\cdot\frac{v^2}{R_E}$$

$$g = \frac{GM_E}{R_E^2}$$

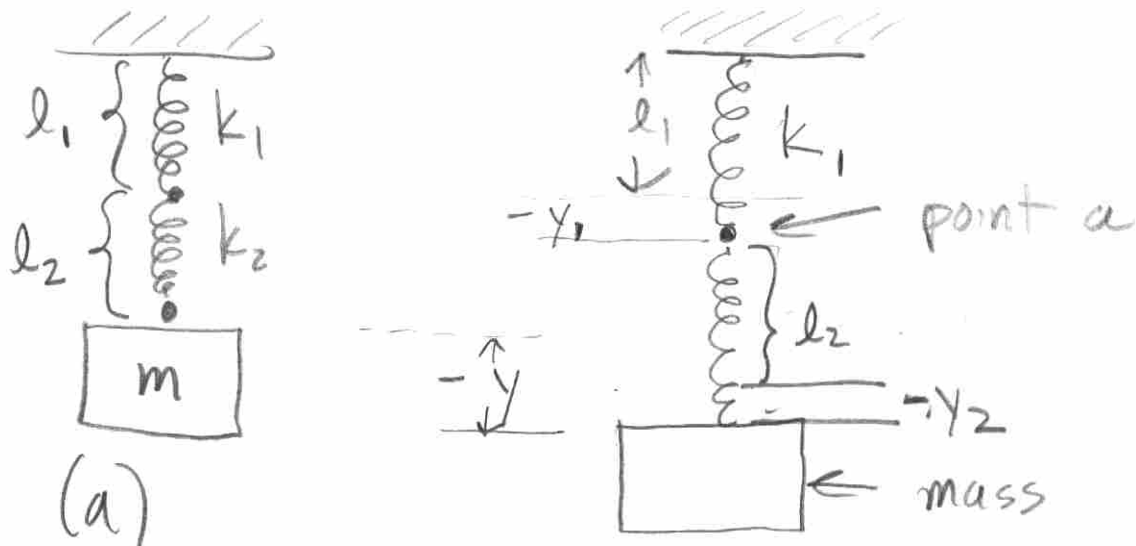
$$v\cdot T = 2\pi R_E$$

$$T = \frac{2\pi R_E}{v}$$

so $v = \sqrt{gR_E}$

$$T = \frac{2\pi R_E}{v} = \frac{2\pi R_E}{\sqrt{gR_E}} = 2\pi\sqrt{\frac{R_E}{g}}$$ <u>same!</u>

7. KK 2.31



(a)

$$-y = -y_1 - y_2$$

point a - no mass,

$$\uparrow +k_1 y_1$$
$$\downarrow -k_2 y_2$$

$$k_1 y_1 - k_2 y_2 = 0$$

$$y_1 = \frac{k_2}{k_1} y_2$$

so $\;y = y_1 + y_2 = \left(\frac{k_2}{k_1} + 1\right) y_2$

but on mass $\quad y_2 = \dfrac{y}{(k_2/k_1 + 1)}$

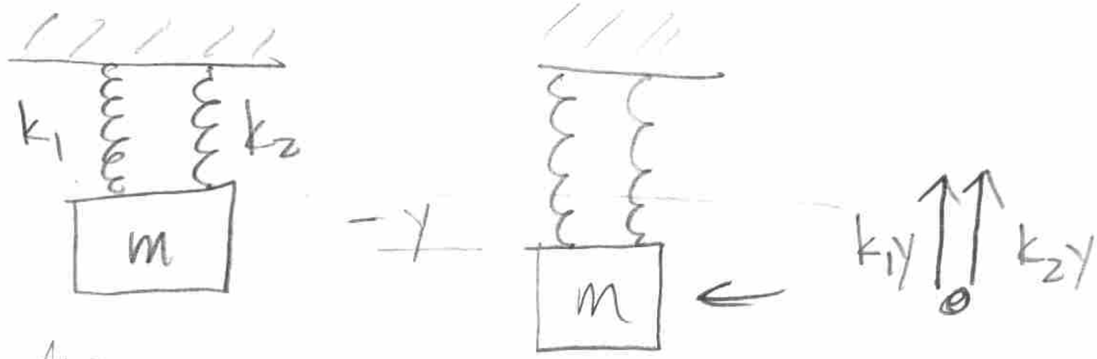
$\uparrow k_2 y_2$

$$F_{springs} = k_2 y_2 = \frac{k_2}{\frac{k_2}{k_1} + 1} y$$

$$F_{sprinks} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}} y = k_a y$$

or $\quad \dfrac{1}{k_a} = \dfrac{1}{k_1} + \dfrac{1}{k_2}, \quad k_a = \dfrac{k_1 k_2}{k_1 + k_2}$

$$\boxed{\omega_a = \sqrt{\frac{k_a}{m}} = \sqrt{\frac{1}{m}\frac{k_1 k_2}{(k_1+k_2)}}}$$



(b)

$$F_{\text{springs}} = (k_1+k_2)y = k_b y$$

$$k_b = k_1 + k_2$$

$$\boxed{\omega_b = \sqrt{\frac{k_b}{m}} = \sqrt{\frac{k_1+k_2}{m}}}$$

Hint: $k_1 = k_2 = k$

$$\omega_a = \sqrt{\frac{1}{m}\frac{k^2}{k+k}} = \sqrt{\frac{k}{2m}} \quad ✓$$

$$\omega_b = \sqrt{\frac{1}{m}\cdot 2k} = \sqrt{2\frac{k}{m}} \quad ✓$$