Phys 20: Mechanics
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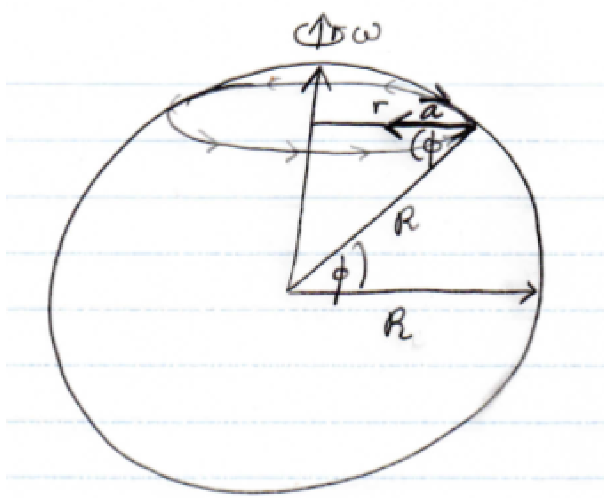
11/1/2010

HW#6 Solutions

1. Diagram:



a) The relation between degrees and radians is

$$\frac{x^\circ}{360^\circ} = \frac{x}{2\pi}$$

so the angle in radians is $\phi = \frac{\pi}{180^\circ} \times 34.41^\circ = 6.006 \times 10^{-1}$.

b) $\cos\phi = \frac{r}{R} \implies r = R\cos\phi$. Given $R = 6371\,km = 6.371 \times 10^3\,km = 6.371 \times 10^6\,m$ and $\phi = 6.006 \times 10^{-1}$, we have $r = 5.256 \times 10^6\,m$.

c) Given: $\omega = \frac{2\pi}{T}$, $T = 8.6164 \times 10^4$s. The speed is

$$v = r\omega = (R\cos\phi)\frac{2\pi}{T} = 3.833 \times 10^2\,m/s\ .$$

d) The point of saying “gravity cancels the normal force” is to say that the only acceleration is due to circular motion, with radius of curvature r. So the acceleration has magnitude

$$|\vec{a}\,| = \frac{v^2}{r} = \frac{(R\cos\phi\,\frac{2\pi}{T})^2}{R\cos\phi} = \left(\frac{2\pi}{T}\right)^2 R\cos\phi = 2.795 \times 10^{-2}\,m/s^2\ .$$

e) See diagram.

f) The percentage of $g = 9.807\, m/s^2$ is

$$\xi_a \equiv \frac{a}{g} \times 100 = 0.2830\,\% \; .$$

g) A scale reads the total force on you. If you put $\mu\, kg$ on it and your acceleration is $a = x\, m/s^2$, then the scale reads $F = (\mu\, kg)(x\, m/s^2) = \mu x\, N$, so that $\mu = \frac{1}{x}(F/N)$. If the total force were due purely to gravity, so that $F = mg$ and $x = g/(m/s^2)$, then $\mu = m/(kg)$ as expected, or in other words μ is your mass in kilograms. However, the force F is not purely due to gravity. Instead, we can parameterize it as $F = mg(1 + \xi_a)$, so that for $x = g/(m/s^2)$ we have

$$\mu = \frac{m}{kg}(1 + \xi_a) \quad \text{where } \xi_a \approx 0.2830\,\% \; .$$

In other words, if the scale claims $\mu = 60$, then your actual mass in units of kilograms is

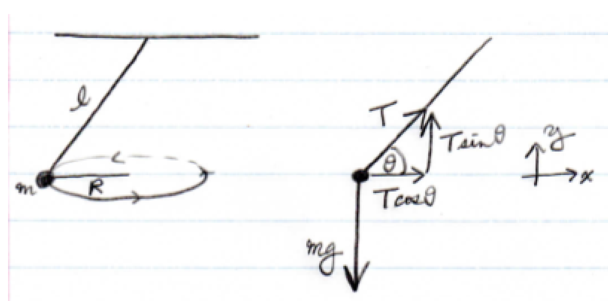
$$\frac{m}{kg} = \frac{\mu}{1 + \xi_a} \approx \frac{60}{1 + 0.2830\%} \approx 59.8 \implies m \approx 59.8\; kg \; .$$

h) The difference in the radius at the equator, R_e, from the radius at the poles, R_p, normalized by the average $\frac{1}{2}(R_e + R_p)$ is

$$\xi_R \equiv \frac{R_e - R_p}{\frac{1}{2}(R_e + R_p)} = 0.3298\,\% \; .$$

We find that $\xi_a \sim \xi_R \sim 0.3\,\%$ roughly. This suggests two things: first, the two effects are related, and second, that to figure out exactly how they are related, we should do a more careful computation. For example, we are at latitude $\phi \sim 34°$, not at the poles.

2. Diagram:



$$\sum F_y = T \sin\theta - mg = 0 \implies T = \frac{mg}{\sin\theta} \; .$$

$$\sum F_x = -T\cos\theta = -m\frac{v^2}{R} \implies T = \frac{mv^2}{R\cos\theta} \; .$$

Setting the two expressions for T equal gives

$$v = \sqrt{\frac{gR}{\tan\theta}} \; .$$

The acceleration $\vec{a} = -\frac{v^2}{R}\hat{r}$ is therefore

$$\vec{a} = -\frac{g}{\tan\theta}\,\hat{r}\ .$$

The magnitude of the acceleration is $|\vec{a}\,| = g/\tan\theta$, and the direction is radially inward: $\hat{a} = -\hat{r}$.

Numbers: $m = 5.3 \times 10^{-2}$ kg, $\ell = 1.4$ m, $R = 0.25$ m. The angle θ is related to the given numbers ℓ and R by $\cos\theta = R/\ell$. That implies $\sin\theta = \sqrt{1-(\cos\theta)^2} = \sqrt{1-R^2/\ell^2}$, which then implies

$$\tan\theta = \sqrt{\frac{\ell^2}{R^2} - 1} \approx 5.51\ .$$

So, we have:

a) The speed of the pebble is

$$v = \sqrt{\frac{gR}{\tan\theta}} = \sqrt{\frac{(9.81\,m/s^2)(0.25\,m)}{5.51}} = 0.67\,m/s\ .$$

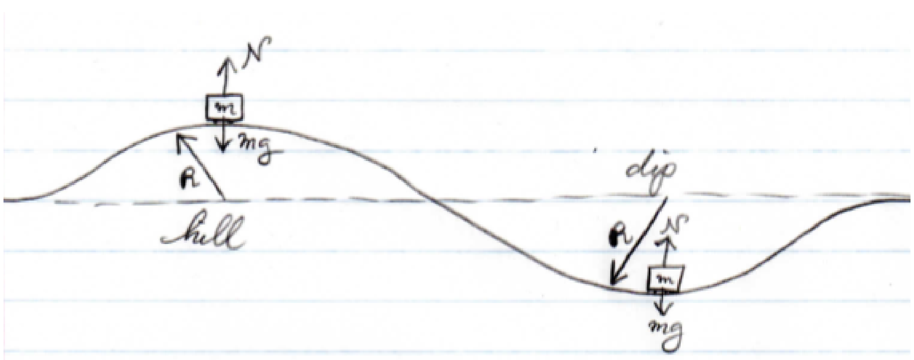
b) The acceleration is

$$\vec{a} = -\frac{v^2}{R}\hat{r} = -\frac{(0.67\,m/s)^2}{0.25\,m}\hat{r} = -(1.8\ m/s^2)\hat{r}\ .$$

c) The tension is

$$T = \frac{mg}{\sin\theta} = \frac{(5.3\times 10^{-2}\,kg)(9.81\,m/s^2)}{\sqrt{1-(0.25\,m)^2/(1.4\,m)^2}} = 0.53\,N\ .$$

3. Diagram:



The sum of forces in the vertical direction when the car is at the top of the hill is

$$\sum F_{\rm up}^{\rm (hill)} = N_{\rm hill} - mg = -m\frac{v_{\rm hill}^2}{R}\quad . \qquad (1)$$

The sum of forces in the vertical direction when the car is at the bottom of the dip is

$$\sum F_{\rm up}^{\rm (dip)} = N_{\rm dip} - mg = +m\frac{v_{\rm dip}^2}{R}\quad . \qquad (2)$$

The plus sign versus the minus sign is the key to this problem. Remember that the acceleration in circular motion is $\vec{a} = -r\dot{\theta}^2\hat{r}$, where $\hat{r}$ is the unit vector that points radially outward *from the center of curvature*. In other words, the radial acceleration points *toward the center of curvature*. For the hill, the center of curvature is below the car, while for the dip the center of curvature is above the car.

Numbers: $R = 250$ m, $mg = 16 \times 10^3$ N.

a) Given: $N_{\rm hill} = \frac{1}{2}mg$. Find: $N_{\rm dip}$.

From equation (1) we know $v_{\rm hill}^2 = \frac{R}{m}(mg - N_{\rm hill})$, and from equation (2) we know $N_{\rm dip} = m(g + v_{\rm dip}^2/R)$. Assuming that the car does not change its speed between leaving the hill and entering the dip, we have $v_{\rm dip} = v_{\rm hill}$, so

$$N_{\rm dip} = mg + (mg - N_{\rm hill}) = (2 - \frac{1}{2})mg = \frac{3}{2}mg\ .$$

b) Find the maximum speed for which the car does not lose contact with the top of the hill.

Remember: "normal force" = "contact force." In other words, "lose contact" means "normal force goes to zero." Take equation (1), set $N_{\rm hill} = 0$ and solve for $v_{\rm hill}$ to get

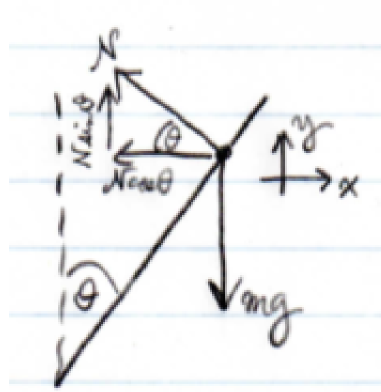
$$v_{\rm hill}^{\rm (MAX)} = \sqrt{gR} = \sqrt{(9.81\,m/s^2)(250\,m)} = 49.5\,m/s\ .$$

c) Given: $v_{\rm dip} = v_{\rm hill}^{\rm MAX}$. Find: $N_{\rm dip}$.

Just plug into equation (2):

$$N_{\rm dip} = mg + m\frac{v_{\rm dip}^2}{R} = mg + mg = 2mg\ .$$

4. Diagram:



Goal: Find R (the distance from the object to the dotted line).

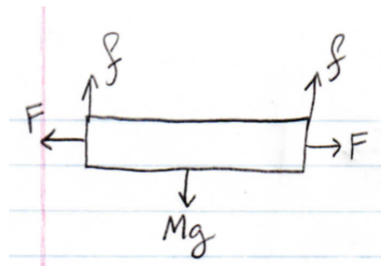
The sums of forces are:

$$\sum F_x : \; -N\cos\theta = -m\frac{v^2}{R} \implies R = \frac{mv^2}{N\cos\theta}$$
$$\sum F_y : \; N\sin\theta - mg = 0 \implies N = \frac{mg}{\sin\theta}$$

So the radius of circular motion is

$$R = \frac{v^2}{g}\,\tan\theta\,.$$

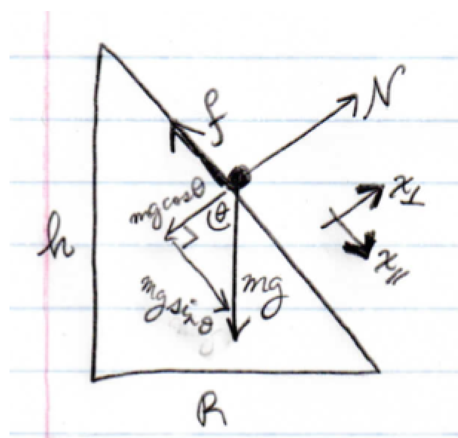
5. Diagram:



The sum of forces in the vertical direction is $2f - Mg$. If this is to equal zero, then $f = \frac{1}{2}Mg$. Since $f = \mu F$, we find that the minimum force required is $F = \frac{1}{2\mu}Mg$.

Numbers: $M = 75$ kg, $\mu = 0.41$. So $F = \frac{1}{2(0.41)}(75\text{ kg})(9.81\text{ m/s}^2) = 897$ N.

6. Diagram:



$$\sum F_\perp : N - mg\,\cos\theta = 0 \implies N = mg\,\cos\theta$$
$$\sum F_\parallel : mg\,\sin\theta - f \underbrace{=}_{\text{WANT}} 0$$

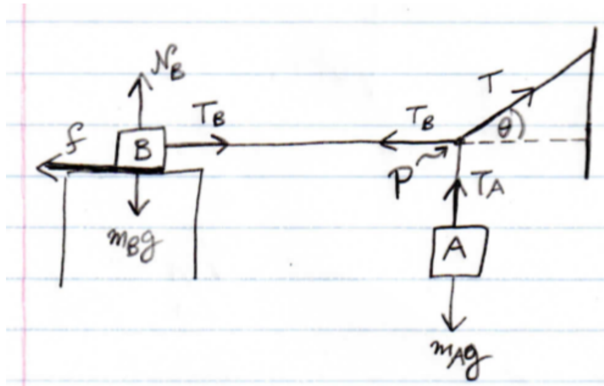
constraint: $f = \mu N$

Using $\sum F_\perp$, we have $f = \mu mg\,\cos\theta$. Plugging that into $\sum F_\parallel$, we find that mg cancels and leaves the result $\mu = \tan\theta$.

But from the triangular cross section of the cone, we know $\tan\theta = h/R$, so $h = \tan\theta R = \mu R$. Therefore the maximum volume is

$$V = \frac{1}{3} h A_{\text{base}} = \frac{1}{3}(\mu R)(\pi R^2) = \frac{1}{3}\mu\pi R^3 \ .$$

7. Diagram:



The forces on block A are:

$$\sum F_{\text{up}} : T_A - m_A g = 0 \implies T_A = m_A g \ .$$
$$\sum F_{\text{right}} : T\cos\theta - T_B = 0 \implies T_B = T\cos\theta \ .$$

The forces on block B are:

$$\sum F_{\text{up}} : N - m_B g = 0 \implies N = m_B g \ .$$
$$\sum F_{\text{right}} : T_B - f = 0 \ , \quad f = \mu N \implies T_B = \mu N = \mu m_B g \ \ \text{(from above)} \ .$$

The forces acting on the point P are

$$\sum F_{\text{up}} : T\sin\theta - T_A = 0 \implies T_A = T\sin\theta \ .$$
$$\sum F_{\text{right}} : T\cos\theta - T_B = 0 \implies T_B = T\cos\theta \ . \leftarrow \text{(redundant)}$$

Now that we have everything we could possibly want to know about the system in equilibrium, let's answer the question. We want to know the maximum weight $W_A = m_A g$ of block A, so we can find the maximum mass of block A and then multiply by g to get a force.

From the forces on point P we know

$$\frac{T_A}{T_B} = \tan\theta$$

and from the forces on blocks A and B we know

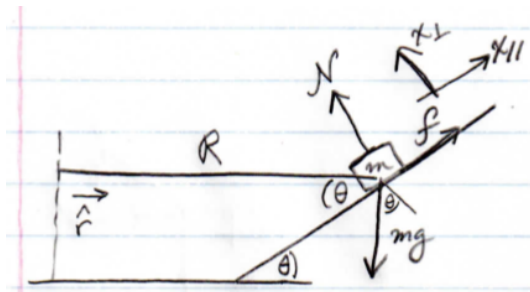
$$\frac{T_A}{T_B} = \frac{m_A g}{\mu m_B g} = \frac{m_A}{\mu m_B} \ .$$

Setting these two equal gives

$$m_A = \mu \tan\theta \, m_B \ .$$

If m_A were greater than this value, then the tension T_B would overcome the force of friction holding block B in place.

8. Diagram:



$$\sum F_\perp : \ N - mgc = ma_\perp \qquad (1)$$
$$\sum F_\parallel : \ f - mgs = ma_\parallel \qquad (2)$$

where $c \equiv \cos\theta$ and $s \equiv \sin\theta$. We also have $f = \mu N$ as usual. As always with circular motion, the acceleration points radially inward, where the radial direction is outward *from the center of curvature* of the motion. Therefore in this problem, the acceleration vector is $\vec{a} = a_r \hat{r}$, where the radial acceleration $a_r = -\frac{v^2}{R}$ is related to the parallel and perpendicular accelerations by

$$a_\parallel = a_r \cos\theta = -\frac{v^2}{R} c \qquad \text{and} \qquad a_\perp = -a_r \sin\theta = +\frac{v^2}{R} s \ .$$

Equations (1) and (2) become

$$N - mgc = +m\frac{v^2}{R} s \qquad (1')$$
$$\mu N - mgs = -m\frac{v^2}{R} c \qquad (2')$$

Equation (1′) gives $N = m(cg + v^2 s/R)$, and equation (2′) gives $N = \frac{m}{\mu}(sg - v^2 c/R)$. Set the two equal, and solve for v^2:

$$\mu(cg + \frac{v^2}{R} s) = gs - \frac{v^2}{R} c$$
$$\frac{1}{R}(\mu s + c) v^2 = g(s - \mu c)$$
$$v^2 = \frac{s - \mu c}{c + \mu s} gR = \frac{\tan\theta - \mu}{1 + \mu \tan\theta} gR$$

This was all assuming that the force of friction is directed up the incline, or in other words that friction opposes a tendency to slide *down* the ramp. The above v should therefore be interpreted as a *minimum* speed required to go around the curve without sliding down the incline:

$$v_{\text{min}} = \sqrt{\frac{\tan\theta - \mu}{1 + \mu\tan\theta}\, gR}\ .$$

What about the opposite case? If the car is trying to slide up the ramp, then friction points down the ramp. If you look at the algebra, all that changes from the previous case is the replacement $\mu \to -\mu$. Therefore, the maximum speed the car can go without sliding up the ramp is

$$v_{\text{max}} = \sqrt{\frac{\tan\theta + \mu}{1 - \mu\tan\theta}\, gR}\ .$$

Note: For the particular choices of parameters $\mu = 1$ and $\tan\theta = 1$, we find $v_{\text{min}} = 0$ and $v_{\text{max}} = \infty$. In other words, the car can go at any speed it wants without sliding either up or down the ramp.