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Phys 20: HW#1 Solutions
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Homework #1

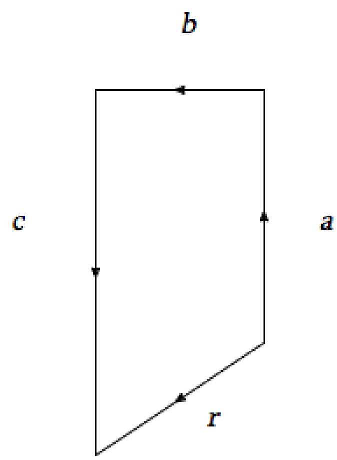
1. Can two vectors having different magnitudes be combined to give a zero vector sum? How about three vectors?

Solution:

Consider two vectors $\vec{a}$ and $\vec{b}$. $\vec{a}+\vec{b}=\vec{0}$ if and only if $\vec{a}=-\vec{b}$, which implies $|\vec{a}|=|\vec{b}|$. So if $|\vec{a}\,|\neq|\vec{b}|$, then $\vec{a}+\vec{b}\neq 0$.

2. A person walks in the following pattern: 3.1km north, then 2.4km west, and finally 5.2km south.
(a) Construct the vector diagram that represents this motion.

Solution:



Labeling: $a=3.1$km, $b=2.4$km, $c=5.2$km. $r=\sqrt{(c-a)^2+b^2}$.

(b) How far and in what direction would a bird fly in a straight line to arrive at the same final point?

Solution:

In polar coordinates (r,θ), the straight line vector from start to finish is $\vec{r}=r(\hat{x}\,\cos(\varphi+\pi)+\hat{y}\,\sin(\varphi+\pi))=-r(\hat{x}\,\cos\varphi+\hat{y}\,\sin\varphi)$. Since $\cos\varphi=b/r$, we have $\sin\varphi=\sqrt{1-(\cos\varphi)^2}=\sqrt{1-b^2/r^2}$. Therefore

$$\vec{r}=-b\hat{x}-\sqrt{r^2-b^2}\,\hat{y}\;.$$

3. KK 1.1: Let $\vec{a} = 2\hat{x} - 3\hat{y} + 7\hat{z}$ and $\vec{b} = 5\hat{x} + \hat{y} + 2\hat{z}$. Compute:
(a) $\vec{a} + \vec{b}$
(b) $\vec{a} - \vec{b}$
(c) $\vec{a} \cdot \vec{b}$
(d) $\vec{a} \times \vec{b}$

Solution:

(a) $\vec{a} + \vec{b} = (2+5)\hat{x} + (-3+1)\hat{y} + (7+2)\hat{z} = 7\hat{x} - 2\hat{y} + 9\hat{z}$.
(b) $\vec{a} - \vec{b} = (2-5)\hat{x} + (-3-1)\hat{y} + (7-2)\hat{z} = -3\hat{x} - 4\hat{y} + 5\hat{z}$.
(c) $\vec{a} \cdot \vec{b} = 2 \cdot 5 + (-3) \cdot 1 + 7 \cdot 2 = 10 - 3 + 14 = 21$.
Note that we have used the orthogonality conditions $\hat{x} \cdot \hat{y} = \hat{y} \cdot \hat{z} = \hat{z} \cdot \hat{x} = 0$ and the normalization conditions $\hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1$.
(d) For $\vec{a} \times \vec{b}$, use the "right-hand rule". That is, $\hat{x} \times \hat{y} = \hat{z}$, $\hat{y} \times \hat{z} = \hat{x}$, $\hat{z} \times \hat{x} = \hat{y}$. Also note that for any two vectors $\vec{\alpha}$ and $\vec{\beta}$, we have $\vec{\alpha} \times \vec{\beta} = -\vec{\beta} \times \vec{\alpha}$ and consequently $\vec{\alpha} \times \vec{\alpha} = 0$ for any vector $\vec{\alpha}$. Now just compute:

$$\begin{aligned}
\vec{a} \times \vec{b} &= (2\hat{x} - 3\hat{y} + 7\hat{z}) \times (5\hat{x} + \hat{y} + 2\hat{z}) \\
&= 2(0 + \hat{x} \times \hat{y} + 2\hat{x} \times \hat{z}) - 3(5\hat{y} \times \hat{x} + 0 + 2\hat{y} \times \hat{z}) + 7(5\hat{z} \times \hat{x} + \hat{z} \times \hat{y} + 0) \\
&= 2(\hat{z} - 2\hat{y}) - 3(-5\hat{z} + 2\hat{x}) + 7(5\hat{y} - \hat{x}) \\
&= (-3 \cdot 2 - 7)\hat{x} + (-2 \cdot 2 + 7 \cdot 5)\hat{y} + (2 + 3 \cdot 5)\hat{z} \\
&= -13\hat{x} + 31\hat{y} + 17\hat{z} \ .
\end{aligned}$$

Alternatively, you can use the antisymmetric tensor to compute the components $(\vec{a} \times \vec{b})_i = \varepsilon_{ijk} a_j b_k \equiv \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} a_j b_k$:

$$\begin{aligned}
(\vec{a} \times \vec{b})_1 &= a_2 b_3 - a_3 b_2 = (-3)(2) - (7)(1) = -13 \\
(\vec{a} \times \vec{b})_2 &= a_3 b_1 - a_1 b_3 = (7)(5) - (2)(2) = 31 \\
(\vec{a} \times \vec{b})_3 &= a_1 b_2 - a_2 b_1 = (2)(1) - (-3)(5) = 17
\end{aligned}$$

so $\vec{a} \times \vec{b} = -13\hat{x} + 31\hat{y} + 17\hat{z}$. From working this out, you can see that that the "right-hand rule" above is equivalent to the conventions $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = +1$, $\varepsilon_{321} = \varepsilon_{213} = \varepsilon_{132} = -1$, and $\varepsilon_{ijk} = 0$ for any two indices ijk equal.

4. KK 1.2: Let $\vec{a} = 3\hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = -2\hat{i} - 3\hat{j} - \hat{k}$. Compute the angle between $\vec{a}$ and $\vec{b}$ and the magnitudes $a \equiv |\vec{a}\,|$ and $b \equiv |\vec{b}\,|$.

Solution:

First the magnitudes: $a^2 = \vec{a} \cdot \vec{a} = 3^2 + 1 + 1 = 11$ so $a = \sqrt{11}$, and $b^2 = \vec{b} \cdot \vec{b} = (-2)^2 + (-3)^2 + (-1)^2 = 4 + 9 + 1 = 14$, so $b = \sqrt{14}$. Now we need $\vec{a} \cdot \vec{b} = 3(-2) + 1(-3) + 1(-1) =$

$-6-3-1=-10$, so the angle γ between $\vec{a}$ and $\vec{b}$ is defined by

$$\cos\gamma = \frac{\vec{a}\cdot\vec{b}}{ab} = \frac{-10}{\sqrt{11}\sqrt{14}} \implies \gamma = \cos^{-1}\left(\frac{-10}{\sqrt{11}\sqrt{14}}\right) \approx 2.51 \approx 147^\circ \, .$$

5. The minute hand of a wall clock measures 10 cm from axis to tip. What is the displacement vector of its tip
(a) from a quarter after the hour to half past, (b) in the next half hour, (c) in the next hour?

Solution:

Let $\vec{a}_0$ be the minute hand at a quarter after the hour, $\vec{a}_1$ be the minute hand at half past the hour, $\vec{a}_2$ be the minute hand at 45 minutes past the hour, and let $\vec{a}_3$ be the minute hand on the hour. In other words, $\vec{a}_0 = a\,\hat{x}$, $\vec{a}_1 = -a\,\hat{y}$, $\vec{a}_2 = -a\hat{x}$ and $\vec{a}_3 = +a\hat{y}$, where $a = 10$ cm. The displacement vector for part (a) is $\vec{d}_a = \vec{a}_1 - \vec{a}_0 = a(-\hat{y}-\hat{x}) = -a(\hat{x}+\hat{y})$. For part (b), the initial position is the same as in part (a), and the final position is 45 minutes past the hour. So the displacement vector for part (b) is $\vec{d}_b = \vec{a}_2 - \vec{a}_0 = a(-\hat{x}-\hat{x}) = -2a\,\hat{x}$. For part (c), the minute hand starts at 15 past and ends at 15 past, so the displacement vector is $\vec{0}$.

6. KK 1.4: Prove that if $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}| = 0$ then $\vec{a}$ and $\vec{b}$ are perpendicular.

Solution:

$|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = a^2+b^2+2\vec{a}\cdot\vec{b}$, where $a \equiv |\vec{a}\,|$ and $b \equiv |\vec{b}\,|$. Similarly, $|\vec{a}-\vec{b}\,|^2 = a^2+b^2-2\vec{a}\cdot\vec{b}$. Setting the two equal implies $2\vec{a}\cdot\vec{b} = -2\vec{a}\cdot\vec{b} \implies \vec{a}\cdot\vec{b} = 0$. Since the angle φ between the vectors $\vec{a}$ and $\vec{b}$ is defined by $\cos\varphi = \vec{a}\cdot\vec{b}/(ab) = 0$, we find $\varphi = 90^\circ$. Therefore $\vec{a}$ and $\vec{b}$ are perpendicular.

7. KK 1.5: Prove that the diagonals of a regular parallelogram are perpendicular.

Solution:

Let $\vec{a}$ and $\vec{b}$ be two adjacent sides of the parallelogram. Then one diagonal is $\vec{a}+\vec{b}$ while the other diagonal is $\vec{a}-\vec{b}$. We want the angle β between the two diagonals, which is defined by

$$\cos\beta = \frac{(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})}{|\vec{a}+\vec{b}||\vec{a}-\vec{b}|} = \frac{a^2-b^2}{|\vec{a}+\vec{b}||\vec{a}-\vec{b}|}$$

where $a \equiv |\vec{a}\,|$ and $b \equiv |\vec{b}\,|$. If $a = b$, then $\cos\beta = 0 \implies \beta = 90^\circ$. Therefore the diagonals $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are perpendicular.

8. KK 1.8: Let $\vec{a} = \hat{\imath}+\hat{\jmath}-\hat{k}$ and $\vec{b} = 2\hat{\imath}-\hat{\jmath}+3\hat{k}$. Find a unit vector $\hat{n}$ perpendicular to both $\vec{a}$ and to $\vec{b}$.

Solution:

Step 1: Compute the vector $\vec{v} \equiv \vec{a} \times \vec{b}$, which is perpendicular to both $\vec{a}$ and to $\vec{b}$.
Step 2: Normalize $\vec{v}$ to 1, or in other words define the unit vector as $\hat{n} \equiv \vec{v}/|\vec{v}\,|$.

The components of $\vec{v}$ are $v_i = \varepsilon_{ijk} a_j b_k$, so:

$$\begin{aligned} v_1 &= a_2 b_3 - a_3 b_2 = (1)(3) - (-1)(-1) = 2 \\ v_2 &= a_3 b_1 - a_1 b_3 = (-1)(2) - (1)(3) = -5 \\ v_3 &= a_1 b_2 - a_2 b_1 = (1)(-1) - (1)(2) = -3 \end{aligned}$$

which means $\vec{v} = 2\hat{\imath} - 5\hat{\jmath} - 3\hat{k}$. The magnitude of $\vec{v}$ is $|\vec{v}\,| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(2)^2 + (-5)^2 + (-3)^2} = \sqrt{4 + 25 + 9} = \sqrt{38}$. Therefore the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is

$$\hat{n} = \frac{1}{\sqrt{38}}(2\hat{\imath} - 5\hat{\jmath} - 3\hat{k}) \; .$$