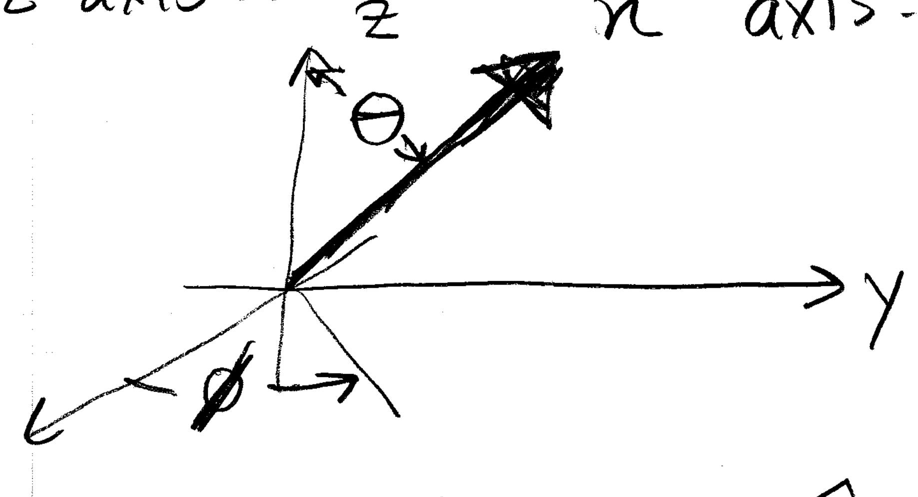
THE BIG FUN

Suppose you don't like the choise of z-axis ... n axis ...



$$\hat{n} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$$

what <u>matrix</u> represents "$\underset{\sim}{S}_n$"

The secret knowledge

Let $$\underset{\sim}{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \equiv \frac{\hbar}{2}\underset{\sim}{\sigma}_z$$

$$\underset{\sim}{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \equiv \frac{\hbar}{2}\underset{\sim}{\sigma}_x$$

$$\underset{\sim}{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar}{2}\underset{\sim}{\sigma}_y$$

$\boxed{\text{CLAIM}}$

$$\underset{\sim}{S}_n = \frac{\hbar}{2}\left[\underset{\sim}{\sigma}_x\sin\theta\cos\phi + \underset{\sim}{\sigma}_y\sin\theta\sin\phi + \underset{\sim}{\sigma}_z\cos\theta\right]$$

Check property #1

$$\underset{\sim}{S}_n^2 \overset{?}{=} \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

<u>note</u>: $\underset{\sim}{S}_z^2 = \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \left(\sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$

$$\underset{\sim}{S}_x^2 = \left(\frac{\hbar}{2}\right)^2 \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

$$\underset{\sim}{\sigma}_x^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\underset{\sim}{S}_y^2 = \left(\frac{\hbar}{2}\right)^2 \left[\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

$$\underset{\sim}{\sigma}_y^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\underset{\sim}{S}_n^2 = \left(\frac{\hbar}{2}\right)^2 \Big\{ \underset{\sim}{\sigma}_x^2 \sin^2\theta\cos^2\phi + \sin^2\theta\sin\phi\cos\phi\,(\underset{\sim}{\sigma}_x\underset{\sim}{\sigma}_y + \underset{\sim}{\sigma}_y\underset{\sim}{\sigma}_x)$$
$$+ \underset{\sim}{\sigma}_y^2 \sin^2\theta\sin^2\phi + \sin\theta\cos\theta\sin\phi\,(\underset{\sim}{\sigma}_y\underset{\sim}{\sigma}_z + \underset{\sim}{\sigma}_z\underset{\sim}{\sigma}_y)$$
$$+ \underset{\sim}{\sigma}_z^2 \cos^2\theta + \sin\theta\cos\theta\cos\phi\,(\underset{\sim}{\sigma}_x\underset{\sim}{\sigma}_z + \underset{\sim}{\sigma}_z\underset{\sim}{\sigma}_y) \Big\}$$

$$1 \cdot \overset{\shortparallel}{} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Pauli Matrices "Anticommute"

<u>all</u> <u>zero</u> $\therefore \underset{\sim}{S}_n^2 = \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

TOPS. 35500

$$\underset{\sim}{S}_n = \frac{\hbar}{2}\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\sin\theta\cos\phi + \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\sin\theta\sin\phi + \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\cos\theta\right]$$

$$= \frac{\hbar}{2}\begin{bmatrix}\cos\theta & \sin\theta\, e^{-i\phi}\\ \sin\theta\, e^{i\phi} & -\cos\theta\end{bmatrix} \Leftarrow \text{"generic" } 2\times 2 \text{ matrix}$$

<u>eigenvectors</u> ?

$$\left(\frac{\hbar}{2}\cos\theta - \lambda\right)\left(-\frac{\hbar}{2}\cos\theta - \lambda\right) - \left(\frac{\hbar}{2}\right)^2\sin^2\theta = 0$$

$$\lambda^2 - \left(\frac{\hbar}{2}\right)^2\left(\cos^2\theta + \sin^2\theta\right) = 0$$

$$\boxed{\lambda = \pm\frac{\hbar}{2}} \quad \text{expected}$$

<u>$\lambda = +\hbar/2$</u> :

$$\frac{\hbar}{2}\begin{bmatrix}\cos\theta - 1 & \sin\theta\, e^{-i\phi}\\ \sin\theta\, e^{i\phi} & -\cos\theta - 1\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} = 0$$

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} \qquad \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\cos\theta - 1 = -2\sin^2\frac{\theta}{2}$$

$$-\cos\theta - 1 = -2\cos^2\frac{\theta}{2}$$

$$\frac{\hbar}{2}\, 2\begin{bmatrix}-\sin^2\frac{\theta}{2} & \sin\frac{\theta}{2}\cos\frac{\theta}{2}\, e^{-i\phi}\\ \sin\frac{\theta}{2}\cos\frac{\theta}{2}\, e^{i\phi} & -\cos^2\frac{\theta}{2}\end{bmatrix}\begin{bmatrix}a_+\\ b_+\end{bmatrix} = 0$$

$$-\sin^2\frac{\theta}{2}\, a_+ + \sin\frac{\theta}{2}\cos\frac{\theta}{2}\, e^{-i\phi}\, b_+ = 0$$

$$\frac{b_+}{a_+} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} e^{i\phi} \qquad \begin{pmatrix} a_+ \\ b_+ \end{pmatrix}_{\hbar/2} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} \end{pmatrix}$$

$\lambda = +\hbar/2$

$$\frac{\hbar}{2} \begin{bmatrix} \cos\theta + 1 & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta + 1 \end{bmatrix} \begin{bmatrix} a_- \\ b_- \end{bmatrix} = 0$$

$$\frac{\hbar}{2} \begin{bmatrix} \cos^2\theta/2 & \sin\frac{\theta}{2}\cos\frac{\theta}{2} e^{-i\phi} \\ \sin\frac{\theta}{2}\cos\frac{\theta}{2} e^{i\phi} & \sin^2\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} a_+ \\ b_- \end{bmatrix} = 0$$

$$\frac{b_-}{a_-} = -\frac{\cos\theta/2}{\sin\theta/2} e^{i\phi}$$

$$\begin{pmatrix} a_- \\ b_- \end{pmatrix}_{-\hbar/2} = \begin{pmatrix} \sin\theta/2\, e^{-i\phi} \\ \cos\theta/2\, e \end{pmatrix}$$

$x: \quad \theta = 90^\circ, \quad \phi = 0$

$$\begin{pmatrix} a_+ \\ b_+ \end{pmatrix}_{\hbar/2} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \qquad \begin{pmatrix} a_- \\ b_- \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

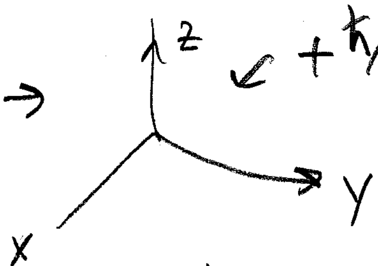
4.2.2.

Spin Eigenstate
in θ, ϕ direction

$$= \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} e^{-i\phi} \\ \sin\frac{\theta}{2} e^{i\phi} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

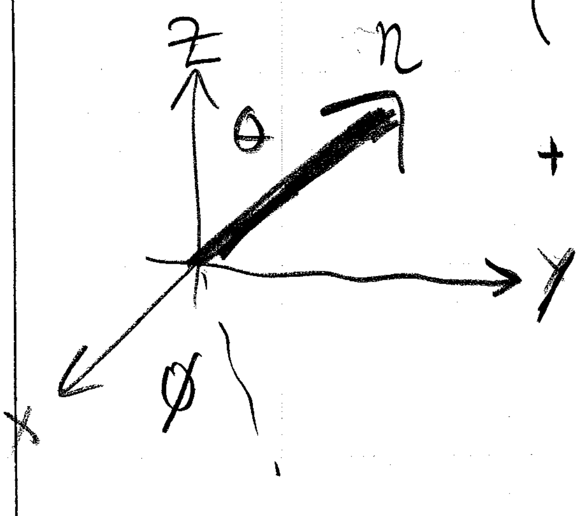
eigenstates in <u>z-basis</u>.

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow$



$+\hbar/2$ in z

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow -\hbar/2$ in z



$+\frac{\hbar}{2}$ along n direction

is $\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} \end{pmatrix}$

in z basis.

$\left.\begin{array}{ll} \cos^2\frac{\theta}{2} & \text{spin up along } z \\ \sin^2\frac{\theta}{2} & \text{spin down along } z \end{array}\right\} !$

subtle

$$\underset{\sim}{U}^{\dagger}(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} e^{-i\phi} \\ \sin\frac{\theta}{2} e^{i\phi} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$\underset{\sim}{U}(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} e^{-i\phi} \\ -\sin\frac{\theta}{2} e^{i\phi} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$\underset{\sim}{U}(\theta)^{\dagger}\underset{\sim}{U}(\theta) = \begin{pmatrix} \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} & \left(\cancel{\cos\frac{\theta}{2}\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\sin\frac{\theta}{2}}\right)e^{-i\phi} \\ \left(\cancel{\sin\frac{\theta}{2}\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)e^{i\phi} & \sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \leftarrow \text{"Unitary"}$$

$\underset{\sim}{U}(\theta)$ above are a representation of SU(2)

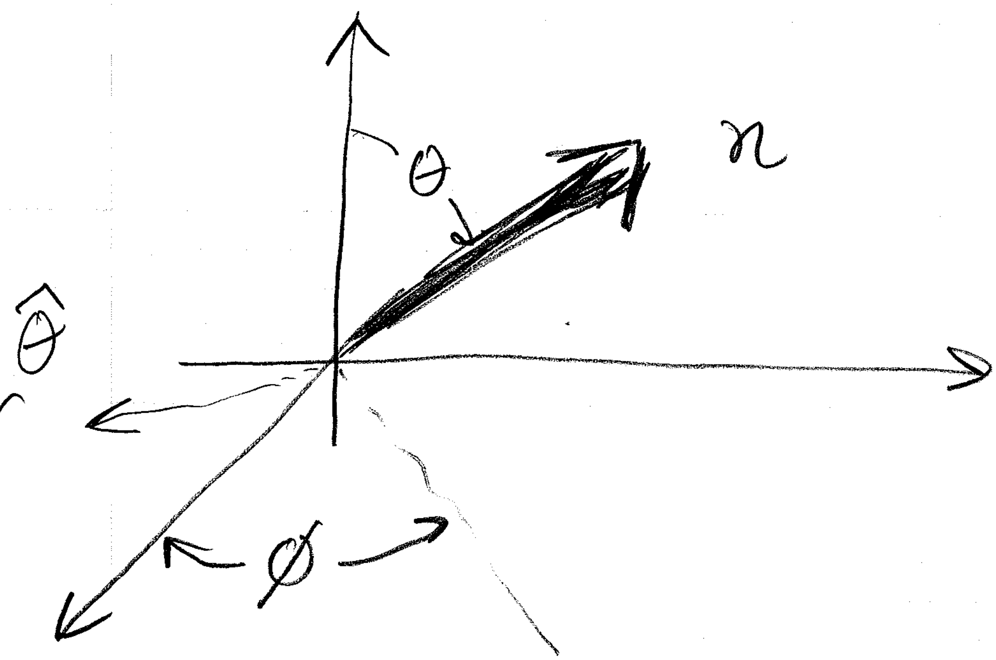
Note:

$$\underset{\sim}{U}(\theta) = \cos\frac{\theta}{2}\cdot\underset{\sim}{\mathbb{1}} - i\sin\frac{\theta}{2}\overbrace{\left(\sin\phi\,\underset{\sim}{\sigma}_x - \cos\phi\,\underset{\sim}{\sigma}_y\right)}^{\text{not so obvious}}$$

$$\sin\theta/2\begin{pmatrix} 0 & -i\sin\phi + \cos\phi \\ -i\sin\phi - \cos\phi & 0 \end{pmatrix}$$

$$= \cos\frac{\theta}{2}\cdot\underset{\sim}{\mathbb{1}} + \begin{pmatrix} 0 & \sin\frac{\theta}{2}e^{-i\phi} \\ -\sin\frac{\theta}{2}e^{i\phi} & 0 \end{pmatrix} \quad ✓$$

<u>Define</u> $\hat{\theta} = \hat{x}\sin\phi - \hat{y}\cos\phi$!



$\hat{\theta}$ is $\perp$ to the ϕ direction!

$$\underset{\sim}{U}(\theta) = \cos\frac{\theta}{2}\cdot\underset{\sim}{\mathbb{1}} - i\sin\frac{\theta}{2}\left[\hat{\theta}\cdot\vec{\underset{\sim}{\sigma}}\right]$$

means $\underset{\sim}{\sigma}_x \sin\phi - \underset{\sim}{\sigma}_y \cos\phi$

$$e^{-i\frac{\alpha}{2}} = \cos\left(\frac{\alpha}{2}\right) - i\sin\left(\frac{\alpha}{2}\right)$$

$$e^{-i\frac{\theta}{2}\hat{\theta}\cdot\vec{\underset{\sim}{\sigma}}} = \cos\frac{\theta}{2}\cdot\underset{\sim}{\mathbb{1}} - i\sin\frac{\theta}{2}\left[\hat{\theta}\cdot\vec{\underset{\sim}{\sigma}}\right].$$

(Spin $\frac{1}{2}$) ... When $\underset{\sim}{H}$ has no preferred direction,

$$[\underset{\sim}{H}, \underset{\sim}{U}^{\dagger}(\theta)] = 0 \quad \forall \theta$$

<u>or</u> $[\underset{\sim}{H}, \underset{\sim}{\sigma}_x] = [\underset{\sim}{H}, \underset{\sim}{\sigma}_y] = [\underset{\sim}{H}, \underset{\sim}{\sigma}_z] = 0$

simultaneous eigenstates.

($u \Leftrightarrow d$) Generalized Spin-$\frac{1}{2}$

$$\begin{pmatrix} u' \\ d' \end{pmatrix} = \underset{\sim}{U}_I^{\dagger}(\theta) \begin{pmatrix} u \\ d \end{pmatrix}$$

for isospin

Your definition of up & down

My definition

(Remember, electric charge neglected)

Sees same physics

$\therefore [\underset{\sim}{H}, \underset{\sim}{U}_I(\theta)] = 0$

simultaneous eigenstates