Scattering and Decays from Fermi's *Golden Rule* including all the $\hbar$'s and c's

(originally by Dirac & Fermi)

References:

Griffins, *Introduction to Quantum Mechanics*, Prentice Hall, 1995.
Perkins, *Introduction to High Energy Physics* 4$^{\text{th}}$ Ed., Cambridge, 2000.
Schiff, *Quantum Mechanics* 2$^{\text{nd}}$ Ed., McGraw Hill, 1955.

Fermi's *Golden Rule*:

Assume the system is described by a Hamiltonian, H:

$$\mathsf{H}\psi = \mathrm{i}\hbar\frac{\partial}{\partial t}\,\psi \tag{1}$$

and that H has the form:

$$\mathsf{H} = \mathsf{H}_0 + \mathsf{H}'$$

where H_0 is the unperturbed Hamiltonian, for which the eigenfunctions ψ_n are known, and H' is the time-dependent perturbation. The eigenfunctions satisfy the following conditions:

$$\mathsf{H}_0\,\psi_n = E_n\psi_n \qquad \langle\psi_a|\psi_b\rangle = \delta_{ab} \tag{2}$$

where the "⟨bra|ket⟩" notation of Dirac implies integration over continuous variables and summation over discrete variables. In this context, δ_{ab} represents a Kronecker delta-function for discrete variables and a Dirac delta-function for continuous variables.

The basic strategy is to express the solution to (1) as a sum over the eigenstates of H_0 with time-dependent coefficients:

$$\psi(t) = \sum_n a_n(t)\,\psi_n\,\mathrm{e}^{-\mathrm{i}E_n t/\hbar} \tag{3}$$

Next, substitute (3) into (1) and use the orthogonality conditions (2) to obtain:

$$\mathrm{i}\hbar\frac{\mathrm{d}a_k(t)}{\mathrm{d}t} = \sum_n \mathsf{H}'_{kn}\, a_n(t)\, \mathrm{e}^{\,\mathrm{i}\omega_{kn}t} \tag{4}$$

where $\mathsf{H}'_{kn} \equiv \langle\psi_k|\mathsf{H}'(t)|\psi_n\rangle$ and $\hbar\,\omega_{kn} \equiv E_k - E_n$. H'_{kn} is often called the "matrix-element" or "transition-amplitude"; it "connects" the states $n \to k$.

Equation (4) is equivalent to the Schrodinger equation (1), but is expressed in terms of the coefficients $a_n(t)$. In simple systems, such as "2-level" systems, (4) can be solved explicitly. In problems involving a continuum of states, scattering for example, (4) is generally solved approximately by a "perturbation expansion." The order-$(p+1)$ approximation is found from the order-(p) solution by:

$$\mathrm{i}\hbar\,\frac{\mathrm{d}}{\mathrm{d}t}\,a_k^{(p+1)}(t) \approx \sum_n \mathsf{H}'_{kn} a_n^{(p)}(t)\, \mathrm{e}^{\,\mathrm{i}\omega_{kn}t} \tag{5}$$

with the "0-th"-order approximation $\mathrm{d}a_k^{(0)}(t)/\mathrm{d}t = 0$, which implies $a_k^{(0)}$ is constant and no transitions occur.

As a first approximation, the system is assumed to be initially in the state m, in which case, $a_n^{(0)}(t) = \delta_{nm}$ and (5) can be integrated to give:

$$\mathrm{i}\hbar\, a_k^{(1)}(t) = \int_{-\infty}^{t} \mathrm{d}t'\, \mathsf{H}'_{km}(t')\, \mathrm{e}^{\,\mathrm{i}\omega_{km}t'} \tag{6}$$

Next, it is assumed that the perturbing force described by H' "turns on" at $t = 0$ and is constant over the interval $0 \le t' \le t$. Equation (6) can then be integrated to give:

$$\mathrm{i}\hbar\, a_k^{(1)}(t) \approx 2\mathsf{H}'_{km}\, \mathrm{e}^{\,\mathrm{i}\omega_{km}t/2} \left(\frac{\sin\omega_{km}t/2}{\omega_{km}}\right)$$

For our purposes, we shall stop the perturbation expansion after the first-order term, in which case $a_k(t) \approx a_k^{(1)}(t)$.

The probability $P_k(t)$ that the system undergoes a transition from state m to state k is:

$$P_k(t) = |a_k(t)|^2 \approx \frac{4\,|\mathsf{H}'_{km}|^2}{\hbar^2}\frac{\sin^2\omega_{km}t/2}{\omega_{km}^2} \tag{7}$$

The mean *rate* for the transition is given by $w_k = P_k(t)/t$. Because of strong peaking in $\frac{1}{t}\frac{\sin^2\omega t/2}{\omega^2}$, near $\omega = 0$ (evident in Figure 1), Equation 7 requires

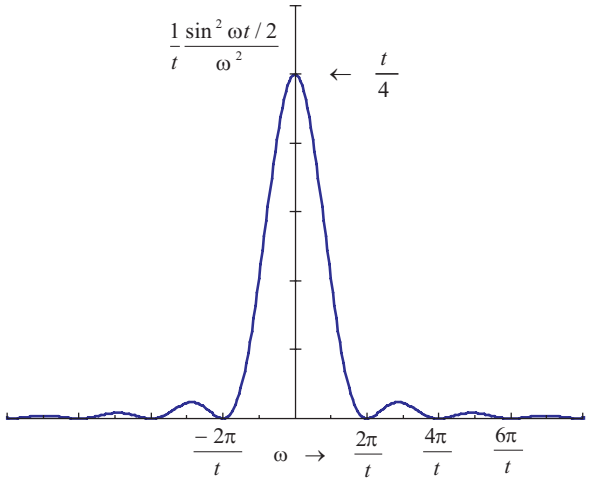

Figure 1: Behavior of the function $g(\omega, t) = \frac{1}{t}\frac{\sin^2 \omega t/2}{\omega^2}$ versus ω. g has the effect of enforcing energy-conservation because in the limit $t \to \infty$, $g \to \frac{\pi}{2}\delta(\omega)$; it explicitly demonstrates the Heisenberg uncertainty relation between energy and time through, for example, the half-width of the peak $\Delta\omega$ and the "lifetime" t of the perturbation: $\Delta\omega\, t \sim \pi$.

that states to which transitions can occur must have $\omega_{km} \approx 0$, forcing energy conservation.

In general, there will be some number of states $\mathrm{d}n$ within an interval $\mathrm{d}\omega_{km}$. The number of possible transition states can be written:

$$\mathrm{d}n = \rho(k)\,\mathrm{d}E_k$$

where $\rho(k) = \mathrm{d}n/\mathrm{d}E_k$ is the "density of states" per unit energy interval near E_k; $\mathrm{d}\omega_{km}$ and $\mathrm{d}E_k$ are related by $\mathrm{d}\omega_{km} = \mathrm{d}E_k/\hbar$. It is expected that $\rho(k)$ and H'_{km} are smoothly varying functions of momentum or energy near the state k.

The physically meaningful quantity is the *total* transition rate to states *near* the state k:

$$W_k = \frac{1}{t}\sum_{k' \text{ near } k} P_{k'}(t)$$

This summation can be replaced by an integral over $\mathrm{d}E_k$:

$$\begin{aligned} W_k &= \frac{1}{t}\int P_{k'}(t)\rho(k')\,\mathrm{d}E_{k'} \\ &= \int \mathrm{d}E_k\,\rho(k)\,\frac{4\,|H'_{km}|^2}{\hbar^2}\,\frac{1}{t}\left(\frac{\sin^2\omega_{km}t/2}{\omega_{km}^2}\right) \\ &= \frac{4}{\hbar}\,|H'_{km}|^2\,\rho(k)\int_{-\infty}^{\infty}\mathrm{d}\omega\frac{1}{t}\frac{\sin^2\omega t/2}{\omega^2} \end{aligned}$$

As can be anticipated from Figure 1, the last integral has the value $\pi/2$ and we arrive at Fermi's "Second Golden Rule":

$$W_k = \frac{2\pi}{\hbar}\,|H'_{km}|^2\,\rho(k) \tag{8}$$

Decays

Equation (8) is used directly to compute decay rates for quantum systems. The mean lifetime τ of the system is related to W_k by $\tau = 1/W_k$. For systems of very short mean lifetimes, the "width" Γ in energy of the state is given by:

$$\Gamma_k = \hbar W_k = 2\pi\,|H'_{km}|^2\,\rho(k)$$

A detailed example: Fermi's theory of nuclear β-decay

The prototypical example of nuclear β-decay is neutron decay $n \to p\,e^-\,\bar{\nu}$. There are many other examples involving nuclei with the same form: $(Z,N) \to (Z+1,N-1)\;e^-\;\bar{\nu}$. On dimensional grounds, the simplest form for the matrix element describing nuclear β-decay is given by Fermi's *ansatz*:

$$H'_{km} = \frac{G_F\,\mathcal{M}}{V} \tag{9}$$

where G_F is a constant—the "Fermi constant". V is the normalization volume used for defining wave functions and $|\mathcal{M}|^2$ describes the overlap of the initial/final nuclear wave functions, a dimensionless quantity expected to be approximately unity.

The energy difference between initial (Z,N) and final $(Z+1,N-1)$ nuclear states is E_0; the system decays to a state of definite energy, but the initial state energy is uncertain to the extent of the finite lifetime, $\Delta E \sim \hbar W$, where W is the decay rate. To compute the rate, the density of possible states $\mathrm{d}n/\mathrm{d}E_0 = \rho(E_0)$ in the region ΔE around E_0 is needed.

We first examine the 3-body kinematics of the problem. The neutrino mass is assumed to be zero and, because typical values of E_0 are in the MeV-range, recoil momenta of all three final-state particles will be typically of order 1 MeV/c. The final-state nucleus (or proton) will, thus, carry negligible kinetic energy ($O \sim 10^{-3}$ MeV/c). Under these assumptions, the decay kinematics are described by:

$$E_0 = E + cq \qquad 0 = \boldsymbol{P} + \boldsymbol{p} + \boldsymbol{q}$$

where E is the electron energy, cq is the neutrino energy, $\boldsymbol{P}$ is the 3-momentum of the decay nucleus, $\boldsymbol{p}$ is the electron 3-momentum and $\boldsymbol{q}$ is the neutrino 3-momentum. The momentum of the decay proton or nucleus is completely determined by the electron and neutrino momenta and, therefore, does not contribute to the density of states.

The density of states is found from the product of the electron and neutrino phase-space volumes:

$$\mathrm{d}n = \frac{V\,\mathrm{d}^3\boldsymbol{p}}{(2\pi\hbar)^3}\,\frac{V\,\mathrm{d}^3\boldsymbol{q}}{(2\pi\hbar)^3}$$

where V is the same normalization volume introduced previously. Notice that the normalization volume used in determining the density of states cancels the V^{-2} factor coming from $|H'_{km}|^2$ and, thus, can be dropped in subsequent formulas. (See footnote on page 7.)

The momentum-space volume elements are given by $\mathrm{d}^3\boldsymbol{p} \to 4\pi p^2 \mathrm{d}p$ and the (unobserved) neutrino momentum volume element can be replaced by $q^2\mathrm{d}q \to (E_0 - E)^2\,\mathrm{d}E_0/c^3$. The density of states is, therefore, given by:

$$\rho(E_0) = \frac{1}{4\pi^4\hbar^6c^3}\,p^2\,(E_0 - E)^2\,\mathrm{d}p$$

and the Golden Rule (8) gives the differential (in the electron momentum p or energy E) decay rate:

$$\mathrm{d}W = \frac{G_F^2}{2\pi^3\hbar^7c^3}\,|\mathcal{M}|^2\,p^2\,(E_0 - E)^2\,\mathrm{d}p \tag{10}$$

Not only do we get the decay rate (by integrating over electron momenta p), but (10) also gives us the *shape* of the decay electron energy spectrum!

Integration of (10) is straightforward; when the electron can be treated as being relativistic ($E \approx cp$), the expression is particularly simple:

$$\int p^2\,(E_0 - E)^2\,\mathrm{d}p \approx \frac{Q_0^5}{30c^3}$$

where $Q_0 = E_0 - m_e c^2$. The total decay rate is thus:

$$W = \frac{1}{\tau} = \frac{G_F^2 \, |\mathcal{M}|^2 \, Q_0^5}{60\pi^3 (\hbar c)^6 \hbar} \tag{11}$$

This result describes vast ranges—15 orders of magnitude—of beta-decay rates in nuclei *and* various "elementary" particles with a common value of G_F and $|\mathcal{M}|^2 \sim 1 - 3$. The Q_0^5-dependence is called "Sargent's Law".

The most accurate determination of G_F* comes from the purely leptonic process of muon decay ($\mu \to e\nu\bar{\nu}$), the rate for which in the fully relativistic calculation has exactly the same form as the β-decay model developed here; the muon decay rate is given by:

$$W_\mu = \frac{G_F^2 \, Q_0^5}{192\pi^3 (\hbar c)^6 \hbar}$$

where, in this case, $Q_0 \approx m_\mu c^2$.

Cross Sections

Consider a 2-body scattering process $a + b \to c + d$ in the center-of-mass frame as depicted in Figure 2. In general, the initial and final momenta p_i, p_f are not the same because particles of different masses may be created in the collision process. The total center-of-mass energy is given by $\sqrt{s} = E_a + E_b = E_c + E_d$.

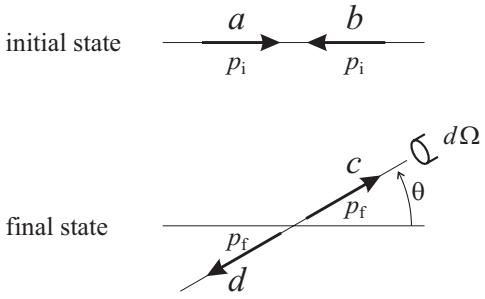

Figure 2: 2-body scattering ($a + b \to c + d$) in the center-of-mass system.

In scattering problems, the transition rate is governed by the "cross section" σ for the process and the flux of initial particles j_i according to:

$$W_f = \mathrm{d}\sigma \, j_i \tag{12}$$

*$G_F = 8.962 \times 10^{-5}$ MeV fm^3

The incident flux is:

$$j_i = \langle \psi_i | \boldsymbol{v}_{\rm op} \cdot \boldsymbol{n} | \psi_i \rangle = v_i / V$$

where $\boldsymbol{v}_{\rm op} \cdot \boldsymbol{n}$ represents the velocity operator along the direction of the collision axis, v_i is the relative speed of a and b in the center-of-mass frame and V is the normalization volume* for wavefunctions ψ.

The relative speed of the initial particles is:

$$v_i = v_a + v_b = c^2 p_i \left(\frac{1}{E_a} + \frac{1}{E_b} \right) = \frac{c^2 p_i \sqrt{s}}{E_a E_b}$$

The density of final states is computed from the phase-space volume of one of the outgoing particles, say particle c; the other particle d is correlated by momentum/energy conservation.

$$\mathrm{d}n = \rho(f) \mathrm{d}\sqrt{s} = \frac{V \, \mathrm{d}^3 \boldsymbol{p}_f}{(2\pi\hbar)^3} \, g_f \tag{13}$$

where g_f is the statistical weight of the final-state spins. For spinless particles, $g_f = 1$; for particles of spins S_c, S_d, respectively, $g_f = (2S_c+1)(2S_d+1)$.

The momentum-space volume is:

$$\mathrm{d}^3 \boldsymbol{p}_f = \mathrm{d}\Omega \, p_f^2 \, \mathrm{d}p_f$$

where $\mathrm{d}\Omega$ is the solid-angle element within which scattered particle c is detected.

Thus, the density of states in scattering problems has the form:

$$\rho(f) = \frac{V \, \mathrm{d}\Omega \, p_f^2 \, g_f}{(2\pi\hbar)^3 \, \frac{\mathrm{d}\sqrt{s}}{\mathrm{d}p_f}}$$

and

$$\frac{\mathrm{d}\sqrt{s}}{\mathrm{d}p_f} = v_c + v_d \equiv v_f = \frac{c^2 p_f \sqrt{s}}{E_c E_d}$$

*The introduction of a "normalization volume" V here and elsewhere in this note may seem arbitrary and obscure. As a practical matter, V is usually set to unity and ignored—the various powers of V that accumulate in a calculation always cancel in the end. V has been kept here as a placeholder to ensure consistent units in all the expressions. Remember, we are trying to keep track of all those c's and $\hbar$'s. The "problem" with V arises from our convention (2) for normalizing "unnormalizable" plane-wave eigenfunctions.

Combining the golden rule (8), the definition of cross section (12) and the density of states (13), we find:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{V^2 \left|\mathsf{H}'_{fi}\right|^2 p_f^2 \, g_f}{4\pi^2 \hbar^4 \, v_i \, v_f}$$

The normalization volume can be "buried" into the definition of the matrix-element M by:

$$\mathsf{H}'_{fi} \equiv \frac{\mathsf{M}_{fi}}{V}$$

The final result is:

$$\begin{aligned} \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{1}{4\pi^2} \frac{|\mathsf{M}_{fi}|^2 \, p_f^2 \, g_f}{\hbar^4 \, v_i \, v_f} \\ &= \frac{1}{4\pi^2} \, |\mathsf{M}_{fi}|^2 \left[\frac{E_a E_b E_c E_d}{(\hbar c)^4 \, s}\right] \left(\frac{p_f}{p_i}\right) g_f \end{aligned} \tag{14}$$

Equation (14) is consistent with the expression for the "scattering amplitude" $f(\boldsymbol{q})$ for non-relativistic, spinless ($g_f = 1$) particles in the Born approximation, as derived in class:

$$f(\boldsymbol{q}) = -\frac{m}{2\pi\hbar^2}\mathsf{M}_{fi} = -\frac{m}{2\pi\hbar^2} \int \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\,\cdot\,\boldsymbol{r}} V(r) \, \mathrm{d}^3\boldsymbol{r} \qquad \boldsymbol{q} = \boldsymbol{p}_f - \boldsymbol{p}_i$$

where $V(r)$ is the scattering potential and m is the reduced-mass of the scattered particle.

Another example: Inverse β-decay or ν-scattering

"Inverse β-decay", the scattering process $\bar{\nu}_e + p \rightarrow n + e^+$, can be described (over certain ranges of energy) by exactly the same simplified model proposed by Fermi to describe β-decay, given above in Equation 9. The result is an isotropic scattering distribution in the center-of-mass frame with a total cross section $\sigma = \int \mathrm{d}\Omega \, (\mathrm{d}\sigma/\mathrm{d}\Omega) = 4\pi \, \mathrm{d}\sigma/\mathrm{d}\Omega$ given by:

$$\sigma = \frac{G_F^2}{\pi} \frac{|\mathcal{M}|^2 p_f^2 \, g_f}{\hbar^4 \, v_i \, v_f} \tag{15}$$

where $|\mathcal{M}|^2$ is, again, dimensionless. The significance of (15) is that rates of processes such as neutron or muon decay can be used to *predict* interaction cross sections for neutrinos because of the linkage through G_F.

For high energy anti-neutrinos ($E \gg m_p c^2$),

$$\sigma \approx \frac{G_F^2\, s}{16\pi(\hbar c)^4} |\mathcal{M}|^2\, g_f$$

For typical interactions in matter, we will assume $|\mathcal{M}|^2 g_f \sim 4$. The target nuclei are assumed to be at rest and the anti-neutrinos have energy $E_{\bar{\nu}}$ in the "lab" frame, in which case $s \approx 2 m_p E_{\bar{\nu}}$. With these assumptions, the numerical value of the neutrino scattering cross section is approximately:

$$\sigma \approx 8 \times 10^{-39} E_{\bar{\nu}}(\text{GeV})\ \text{cm}^2$$

which is tiny! For example, the mean free path for 10 GeV anti-neutrinos in material having the same average density as the earth is about $\lambda \sim 4 \times 10^{12}$ cm, which is about 60,000 times the earth's radius.