Conclusion:

the space spanned by $(2j_1+1)\cdot(2j_2+1)$ "product" states

$|j_1\, m_1, j_2\, m_2\rangle$ (eigenkets of $J_{1z}+J_{2z}$ with e.v. $(m_1+m_2)\hbar$; $\vec{J}_1^{\,2}$ e.v. $j_1(j_1+1)\hbar^2$, $\vec{J}_2^{\,2}$ e.v. $j_2(j_2+1)\hbar^2$)

is also spanned by the $(2j_1+1)\cdot(2j_2+1)$ eigenkets of $(\vec{J}_1+\vec{J}_2)^2$ with e.v. $j(j+1)\hbar^2$ $|j_1-j_2|\le j < j_1+j_2$

$|j\, m\rangle$ $m = m_1+m_2$

$j \longrightarrow$; $m \downarrow$

j_1+j_2	j_1+j_2-1	- - - - -	$\lvert j_1-j_2\rvert$
$\lvert j_1+j_2, j_1+j_2\rangle$			
$\lvert j_1+j_2, j_1+j_2-1\rangle$	$\lvert j_1+j_2-1, j_1+j_2-1\rangle$		
$\lvert j_1+j_2, j_1+j_2-2\rangle$	$\lvert j_1+j_2-1, j_1+j_2-2\rangle$		$\lvert\, \lvert j_1-j_2\rvert, \lvert j_1-j_2\rvert\rangle$
$\vdots$	$\vdots$		$\vdots$
			$\lvert\, \lvert j_1-j_2\rvert, -\lvert j_1-j_2\rvert\rangle$
$\lvert j_1+j_2, -(j_1+j_2-2)\rangle$	$\lvert j_1+j_2-1, -(j_1+j_2-2)\rangle$		
$\lvert j_1+j_2, -(j_1+j_2-1)\rangle$	$\lvert j_1+j_2-1, -(j_1+j_2-1)\rangle$		
$\lvert j_1+j_2, -(j_1+j_2)\rangle$			

suppose $j_1 = 3$ $j_2 = 2$ $\rightarrow$ $(2\cdot3+1)\cdot(2\cdot2+1) = 35$ product states.

$j=5$	$j=4$	$j=3$	$j=2$	$j=1$
$\lvert 5,5\rangle$				
$\lvert 5,4\rangle$	$\lvert 4,4\rangle$			
$\lvert 5,3\rangle$	$\lvert 4,3\rangle$	$\lvert 3,3\rangle$		
			$\lvert 2,2\rangle$	$\lvert 1,1\rangle$
$\vdots$	$\vdots$	$\vdots$		
			$\lvert 2,-2\rangle$	$\lvert 1,-1\rangle$
$\lvert 5,-3\rangle$	$\lvert 4,-3\rangle$	$\lvert 3,-3\rangle$		
$\lvert 5,-4\rangle$	$\lvert 4,-4\rangle$			
$\lvert 5,-5\rangle$				
$2\cdot5+1 = 11$	$2\cdot4+1=9$	$2\cdot3+1=7$	$2\cdot2+1=5$	3

$11+9+7+5+3$
$= 20+12+3$
$= 35$ "total j" states.

or $3 \otimes 2 = 5 \oplus 4 \oplus 3 \oplus 2 \oplus 1$

How does one relate the product states $|j_1 m_1, j_2 m_2\rangle$ to the eigenstates of $(\vec{\underset{\sim}{J}}_1 + \vec{\underset{\sim}{J}}_2)^2$, $|j\, m\rangle$? (properly, $|j m, j_1 j_2\rangle$ but we omit j_1 and j_2)

<u>Two</u> states $|j\, m\rangle$ are really easy to relate to $|j_1 m_1, j_2 m_2\rangle$. These are the states where $j = j_1 + j_2$ and $|m| = j_1 + j_2$; where m is maximal.

$$|\overset{j}{j_1+j_2}, \underbrace{\overset{m}{j_1+j_2}}_{\text{maximal}}\rangle = 1 \cdot |\underbrace{\overset{j_1}{j_1}\overset{m_1}{j_1}, \overset{j_2}{j_2}\overset{m_2}{j_2}}_{\text{maximal too}}\rangle$$

$$|j_1+j_2, \underbrace{-(j_1+j_2)}_{\text{minimal}}\rangle = 1 \cdot |\underbrace{j_1\, {-j_1}, j_2\, {-j_2}}_{\text{minimal too}}\rangle$$

no other ways.

convention (Condon Shortley): use 1, not $e^{i\theta}$, $\theta \neq 0$.

Now we can get more by using the lowering operators (or raising operators)

$$\underset{\sim}{J}_- |j_1+j_2, j_1+j_2\rangle = (\underset{\sim}{J}_{1-} + \underset{\sim}{J}_{2-}) |j_1 j_1, j_2 j_2\rangle$$

$$\underset{\sim}{J}_- |j\, m\rangle = \hbar [(j+m)(j-m+1)]^{1/2} |j\, m-1\rangle$$

$$\underset{\sim}{J}_- |j_1+j_2, j_1+j_2\rangle = \hbar [2(j_1+j_2) \times 1]^{1/2} |j_1+j_2, j_1+j_2-1\rangle$$

$$(\underset{\sim}{J}_{1-} + \underset{\sim}{J}_{2-}) |j_1 j_1, j_2 j_2\rangle = \hbar [(2j_1)^{1/2}] |j_1 (j_1-1), j_2 j_2\rangle + \hbar (2j_2)^{1/2} |j_1 j_1, j_2 (j_2-1)\rangle$$

$$\text{or } |j_1+j_2, j_1+j_2-1\rangle = \underbrace{\left[\frac{j_1}{j_1+j_2}\right]^{1/2}}_{\text{a Clebsch-Gordan coefficient}} |j_1 (j_1-1), j_1 j_2\rangle + \underbrace{\left[\frac{j_2}{j_1+j_2}\right]^{1/2}}_{\text{another C-G coefficient}} |j_1 j_1, j_2 (j_2-1)\rangle$$

more specifically: $3 \otimes 2$

$$|5\,5\rangle = |3\,3, 2\,2\rangle$$
$$m = m_1 + m_2$$

$$\underset{\sim}{J}_-|5\,5\rangle = \hbar\left[(5+5)(5-5+1)\right]^{1/2}|5\,4\rangle = \hbar\sqrt{10}\,|5\,4\rangle$$

$$\underset{\sim}{J}_{1-}|3\,3, 2\,2\rangle = \hbar\sqrt{6}\,|3\,2, 2\,2\rangle \qquad \underset{\sim}{J}_{2-}|3\,3, 2\,2\rangle = \hbar\sqrt{4}\,|3\,3, 2\,1\rangle$$

so $\quad \underset{\sim}{J}_-|5\,5\rangle = (\underset{\sim}{J}_{1-} + \underset{\sim}{J}_{2-})|3\,3, 2\,2\rangle$

$$\hbar\sqrt{10}\,|5\,4\rangle = \hbar\sqrt{6}\,|3\,2, 2\,2\rangle + \hbar\sqrt{4}\,|3\,3, 2\,1\rangle$$

$$\boxed{|5\,4\rangle = \sqrt{\tfrac{3}{5}}\,|3\,2, 2\,2\rangle + \sqrt{\tfrac{2}{5}}\,|3\,3, 2\,1\rangle \qquad 4 \quad 2+2=4 \quad 3+1=4}$$

could keep going to $|5, 3\rangle$, $|5\,2\rangle$, etc.
The "action" is on the right hand side

$$\underset{\sim}{J}_-|5\,4\rangle = \hbar\left[(5+4)(5-4+1)\right]^{1/2}|5\,3\rangle = \hbar\,[9\cdot 2]^{1/2}|5\,3\rangle$$

$$(\underset{\sim}{J}_{1-} + \underset{\sim}{J}_{2-})|3\,2, 2\,2\rangle = \hbar\left(\left[(3+2)(3-2+1)\right]^{1/2}|3\,1, 2\,2\rangle + \left[(2+2)(2-2+1)\right]^{1/2}|3\,2, 2\,1\rangle\right)$$
$$= \hbar\left([5\cdot 2]^{1/2}|3\,1, 2\,2\rangle + [4]^{1/2}|3\,2, 2\,1\rangle\right)$$

$$(\underset{\sim}{J}_{1-} + \underset{\sim}{J}_{2-})|3\,3, 2\,1\rangle = \hbar\left([3\cdot 2]^{1/2}|3\,2, 2\,1\rangle + \overset{(2+1)(2-1+1)}{[3\cdot 2]^{1/2}}|3\,3, 2\,0\rangle\right)$$

$$\sqrt{9\cdot 2}\,|5\,3\rangle = \sqrt{\tfrac{3}{5}}\sqrt{5\cdot 2}\,|3\,1, 2\,2\rangle + \left(\sqrt{\tfrac{3}{5}}\cdot\sqrt{2\cdot 2} + \sqrt{\tfrac{2}{5}}\cdot\sqrt{3\cdot 2}\right)|3\,2, 2\,1\rangle + \sqrt{\tfrac{2}{5}}\sqrt{3\cdot 2}\,|3\,3, 2\,0\rangle$$

$$\boxed{|5\,3\rangle = \sqrt{\tfrac{1}{3}}\,|3\,1, 2\,2\rangle + 2\sqrt{\tfrac{3\cdot 2}{9\cdot 5}}\,|3\,2, 2\,1\rangle + \sqrt{\tfrac{3\cdot 2}{9\cdot 5}}\,|3\,3, 2\,0\rangle}$$

$$\frac{1}{3} + \frac{4\cdot 3\cdot 2}{45} + \frac{3\cdot 2}{45} = \frac{15 + 24 + 6}{45} = 1$$

et cetera ... can get all $|5\,m\rangle$; generally $|j_1 + j_2, m\rangle$

---

to get to different values of total J, use orthogonality.

$|j_1+j_2-1, j_1+j_2-1\rangle$ is orthogonal to

$$|j_1+j_2, j_1+j_2-1\rangle = \left[\frac{j_1}{j_1+j_2}\right]^{1/2} |j_1(j_1-1), j_2 j_2\rangle + \left[\frac{j_2}{j_1+j_2}\right]^{1/2} |j_1 j_1, j_2(j_2-1)\rangle$$

$$|j_1+j_2-1, j_1+j_2-1\rangle = -\left[\frac{j_2}{j_1+j_2}\right]^{1/2} |j_1(j_1-1), j_2 j_2\rangle + \left[\frac{j_1}{j_1+j_2}\right]^{1/2} |j_1 j_1, j_2(j_2-1)\rangle$$

so $\boxed{|44\rangle = -\sqrt{\tfrac{2}{5}}\,|32,22\rangle + \sqrt{\tfrac{3}{5}}\,|33,21\rangle}$

could now use lowering operators to get $|43\rangle$, $|42\rangle$, etc. Then could get $|33\rangle$ by requiring that it be orthogonal to $|53\rangle$ and $|43\rangle$.

Now that you know how to derive the C-G coefficients yourself, let's develop how to use tables + programs.

$$\underbrace{|jm, j_1 j_2\rangle}_{\substack{\text{eigenket of}\\ (\underset{\sim}{\vec{J}}_1+\underset{\sim}{\vec{J}}_2)^2 \to \hbar^2 j(j+1)\\ (J_{1z}+J_{2z}) \to \hbar m\\ \vec{J}_1^2 \to \hbar^2 j_1(j_1+1)\ \ \vec{J}_2^2 \to \hbar^2 j_2(j_2+1)}} = \sum_{m_1, m_2} \underbrace{|j_1 m_1, j_2 m_2\rangle}_{\substack{\text{product ket}\\ \text{not an eigenket}\\ \text{of } (\vec{J}_1+\vec{J}_2)^2}} \underbrace{\langle j_1 m_1, j_2 m_2 | jm, j_1 j_2\rangle}_{\substack{\text{Clebsch}\\ \text{Gordan}\\ \text{coefficient}}}$$

Properties of $\langle j_1 m_1, j_2 m_2 | jm, j_1 j_2\rangle$

1. unless $|j_1 - j_2| \le j \le j_1 + j_2$ $\langle j_1 m_1, j_2 m_2 | jm, j_1 j_2\rangle = 0$
   (we knew this already)

2. unless $m_1 + m_2 = m$, $\langle j_1 m_1, j_2 m_2 | jm, j_1 j_2\rangle = 0$
   (since $\underset{\sim}{J}_{1z} + \underset{\sim}{J}_{2z} = \underset{\sim}{J}_z$)

3. all real by convention

4. Phase convention: $\langle j_1 j_1, j_2 (j-j_1) | j j \rangle$ is positive

5. $\langle j_1 m_1, j_2 m_2 | j m \rangle = (-1)^{j_1+j_2-j} \langle j_1 (-m_1), j_2 (-m_2) | j (-m) \rangle$

→ saves time for the "negative" m

→ if $j_1+j_2=j$, all are positive (obvious)

→ sign flipping starts at $j=j_1+j_2-1$

There are tables! So, suppose you want to combine $j_1=\frac{3}{2}$ $j_2=1$. Use the table..

know $j=5/2, 3/2, 1/2$ possible

suppose you want $|\frac{3}{2}\ {-\frac{1}{2}}\rangle$
$\quad j \quad m$

$$\underset{j\ \ m}{|\tfrac{3}{2}\ {-\tfrac{1}{2}}\rangle} = a\,\underset{m_1+m_2=-\frac{1}{2}}{|\tfrac{1}{2}\ {-1}\rangle} + b\,\underset{m_1\ m_2}{|{-\tfrac{1}{2}}\ 0\rangle} + c\,\underset{m_1\ m_2}{|{-\tfrac{3}{2}}\ 1\rangle}$$

$\Leftarrow$ | → this side label $|m_1\ m_2\rangle$

go to table:
find $\frac{3}{2}\times 1$

m_1	m_2	$j=\frac{3}{2}$, $m=-\frac{1}{2}$
$+1/2$	-1	$8/15$
$-1/2$	0	$-1/15$
$-3/2$	$+1$	$-2/15$

means: $a=\sqrt{\frac{8}{15}}$ $\quad b=-\sqrt{\frac{1}{15}}$ $\quad c=-\sqrt{\frac{2}{5}}$

a.k.a. $\langle \frac{3}{2}\frac{1}{2}, 1 -1 | \frac{3}{2} -\frac{1}{2}\rangle$ $\quad \langle \frac{3}{2} -\frac{1}{2}, 1 0 | \frac{3}{2} -\frac{1}{2}\rangle$ $\quad \langle \frac{3}{2} -\frac{3}{2}, 1 1 | \frac{3}{2} -\frac{1}{2}\rangle$

Inversion of Clebsch-Gordan

You can view the C-G coefficients as elements of a unitary matrix. This means that if you want to express a product ket (like $|\frac{3}{2}\frac{1}{2}, 1 -1\rangle$) as a superposition of eigenkets of total angular momentum, you can read horizontally across the tables (transpose!).

$m_1\ m_2$	$J=5/2$, $m=-1/2$	$3/2$, $-1/2$	$1/2$, $-1/2$
$+\frac{1}{2}\ -1$	$3/10$	$-8/15$	$1/6$

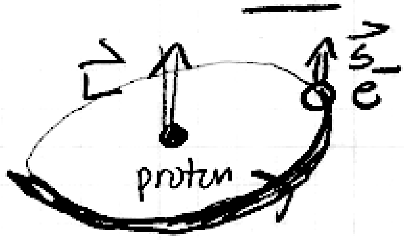
means

$$\underset{m_1\ m_2}{|{+\tfrac{1}{2}}\ {-1}\rangle} = \sqrt{\tfrac{3}{10}}\,\underset{j\ \ m}{|\tfrac{5}{2}\ {-\tfrac{1}{2}}\rangle} - \sqrt{\tfrac{8}{15}}\,\underset{j\ \ m}{|\tfrac{3}{2}\ {-\tfrac{1}{2}}\rangle} + \sqrt{\tfrac{1}{6}}\,\underset{j\ \ m}{|\tfrac{1}{2}\ {-\tfrac{1}{2}}\rangle}$$

aka $|\frac{3}{2}\frac{1}{2}, 1 -1\rangle$

Spin-Orbit Coupling

electron in orbit has $\vec{L}$ and $\vec{S}$ } product states actually not eigenstates of full $\underset{\sim}{H}$, because $\underset{\sim}{H}$ has a piece $\propto \underset{\sim}{\vec{L}} \cdot \underset{\sim}{\vec{S}}$. (spin-orbit coupling)



means: eigenkets of $\underset{\sim}{H}$ will be eigenkets of $\underset{\sim}{J}^2 = (\underset{\sim}{\vec{L}} + \underset{\sim}{\vec{S}})^2$

eigenvalues? $J = L \pm 1/2$

Spectroscopic notation: S P D F G H
$L=$ 0 1 2 3 4 5 ...

${}^{2S+1}L_J$ $\quad S=$ total spin (sum of electrons) spins...

$\hookrightarrow$ ${}^1S_0$ = helium ground state $\quad L=0;\ S=0$ so $J=0$

Lithium <u>could</u> have $S = 3/2$ or $= 1/2$
(3 electrons.)

$L = 1$ so J could be: $5/2, 3/2, 1/2$
(ground state).

Lithium could be $^{2\cdot\frac{3}{2}+1}P_{5/2} = {}^4P_{5/2}$

$L = 1$

or $^4P_{3/2}$, $^4P_{1/2}$, $^2P_{5/2}$, $^2P_{3/2}$, $^2P_{1/2}$ etc.

Your old table of charmonium, bottomonium had some of these labels

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Skip chapter 16 (will possibly return)

Time Independent Perturbation Theory.

<u>Idea</u>: $\underset{\sim}{H} = \underset{\sim}{H}^0 + \underset{\sim}{H}'$

$\underset{\sim}{H}^0$: dominant

$\underset{\sim}{H}'$: small compared to $\underset{\sim}{H}^0$.

assume $\underset{\sim}{H}^0$ has a basis of eigenkets $|n^0\rangle$ (non-degenerate).

like: $|1^0\rangle, |2^0\rangle, |3^0\rangle$ — the ^{0} superscript is a <u>label</u>, <u>not a power</u>.

$$\underset{\sim}{H}^0 |n^0\rangle = E_n^0$$

then, we imagine expanding the <u>exact</u> eigenkets $|n\rangle$

$$\underset{\sim}{H} |n\rangle = E_n |n\rangle$$

in a power series in, crudely, $\underset{\sim}{H}'/\underset{\sim}{H}^0$.