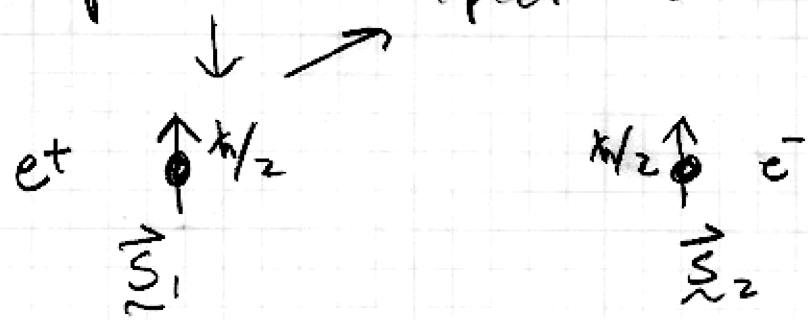
Addition of Angular Momentum

Only <u>total</u> angular momentum is conserved

for example:

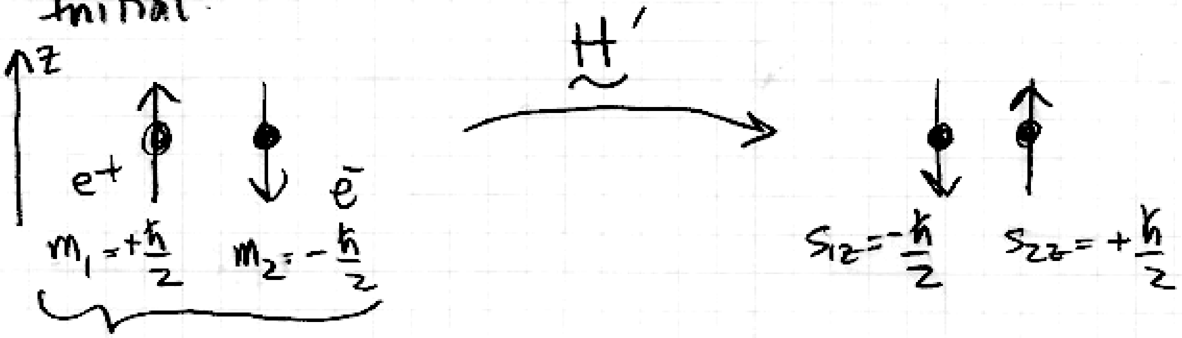
classical picture... expect $0 < |\vec{S}| = |\vec{S}_1 + \vec{S}_2| \leq \hbar$



Quantum Mechanically, what will this system be? Total angular momentum 1 or 0, or a little of both?

Why does it matter? There <u>are</u> interactions that can cause spontaneous spin-flip:

Initial:



$S_z = S_{1z} + S_{2z}$ and here

eigenvalue $= 0$ here

but is the total spin 1 or 0?

$|s_1 m_1\rangle =$ either $|\tfrac{1}{2}\,\tfrac{1}{2}\rangle$ or $|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle$

$|\uparrow_1\rangle$ or $|\downarrow_1\rangle$

Two vectors span "particle #1's" space

Particle #2 ?

$|s_2\, m_2\rangle \rightarrow |\tfrac{1}{2}\, \tfrac{1}{2}\rangle$ or $|\tfrac{1}{2}\, \tfrac{-1}{2}\rangle$

$|\uparrow_2\rangle$ or $|\downarrow_2\rangle$

System with two particles in it spanned by

$|s_1 m_1\rangle \otimes |s_2\, m_2\rangle = |s_1 m_1, s_2 m_2\rangle$ (total of 4 vectors)

component eigenvalues must be additive.

$$\underset{\sim}{S}_{1z}|s_1 m_1, s_2 m_2\rangle = m_1\hbar\, |s_1 m_1, s_2 m_2\rangle$$
$$\underset{\sim}{S}_{2z}|s_1 m_1, s_2 m_2\rangle = m_2\hbar\, |s_1 m_1, s_2 m_2\rangle$$

$$\underset{\sim}{S}_z|s_1 m_1, s_2 m_2\rangle = (m_1 + m_2)\hbar\, |s_1 m_1, s_2 m_2\rangle$$

$$\underset{\sim}{S}_z = \underset{\sim}{S}_{1z} + \underset{\sim}{S}_{2z}$$

How about $\underset{\sim}{S}^2$? $= (\vec{\underset{\sim}{S}}_1 + \vec{\underset{\sim}{S}}_2)^2$

$= \underset{\sim}{S}_1^2 + \underset{\sim}{S}_2^2 + 2\vec{\underset{\sim}{S}}_1 \cdot \vec{\underset{\sim}{S}}_2$ (commute; order un important)

$\underset{\sim}{S}_1^2 |s_1 m_1, s_2 m_2\rangle = \hbar^2 s_1(s_1+1)|s_1 m_1, s_2 m_2\rangle$ eigenkets

$\underset{\sim}{S}_2^2 |s_1 m_1, s_2 m_2 = \hbar^2 s_2(s_2+1)|s_1 m_1, s_2 m_2\rangle$

the sticky piece is $\vec{\underset{\sim}{S}}_1 \cdot \vec{S}_2$. This is not even diagonal in the basis we are using. BTW, to save writing:

$|\tfrac{1}{2}\, \tfrac{1}{2}\rangle_1 = |\uparrow_1\rangle \quad |\tfrac{1}{2}\, \tfrac{1}{2}\rangle_2 = |\uparrow_2\rangle$

"product state" $|\uparrow_1\rangle|\uparrow_2\rangle = |\uparrow_1\uparrow_2\rangle$

others $|\uparrow_1\downarrow_2\rangle$, $|\downarrow_1\uparrow_2\rangle$, $|\downarrow_1\downarrow_2\rangle$

Study the dot product operator

$$\vec{\underset{\sim}{S}}_1 \cdot \vec{\underset{\sim}{S}}_2 \equiv \underset{\sim}{S}_{1x}\underset{\sim}{S}_{2x} + \underset{\sim}{S}_{1y}\underset{\sim}{S}_{2y} + \underset{\sim}{S}_{1z}\underset{\sim}{S}_{2z}$$

operates in #1's space

#2's space

A natural way to represent this operator is in the space of products of eigenstates of $\underset{\sim}{S}_{1z}$ + $\underset{\sim}{S}_{1z}$:

$\#1 = |\uparrow_1 \uparrow_2\rangle$ $\quad \#2 = |\uparrow_1 \downarrow_2\rangle$ $\quad \#3 = |\downarrow_1 \uparrow_2\rangle$ $\quad \#4 = |\downarrow_1 \downarrow_2\rangle$

take one matrix element:

$$\langle \uparrow_1 \uparrow_2 | \vec{\underset{\sim}{S}}_1 \cdot \vec{\underset{\sim}{S}}_2 | \uparrow_1 \uparrow_2 \rangle$$

$$\equiv \langle \uparrow_1 | \underset{\sim}{S}_{1x} | \uparrow_1 \rangle \langle \uparrow_2 | \underset{\sim}{S}_{2x} | \uparrow_2 \rangle + \langle \uparrow_1 | \underset{\sim}{S}_{1y} | \uparrow_1 \rangle \langle \uparrow_2 | \underset{\sim}{S}_{2y} | \uparrow_2 \rangle$$
$$+ \langle \uparrow_1 | \underset{\sim}{S}_{1z} | \uparrow_1 \rangle \langle \uparrow_2 | \underset{\sim}{S}_{2z} | \uparrow_2 \rangle$$

$$\langle \uparrow_1 | \underset{\sim}{S}_{1x} | \uparrow_1 \rangle = (1 \ 0) \times \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}(1 \ 0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$= \langle \uparrow_2 | \underset{\sim}{S}_{2x} | \uparrow \rangle = \langle \uparrow_1 | \underset{\sim}{S}_{1y} | \uparrow_1 \rangle = \langle \uparrow_2 | \underset{\sim}{S}_{2y} | \uparrow_2 \rangle$$

$$\langle \uparrow_1 | \underset{\sim}{S}_{1z} | \uparrow_1 \rangle = \frac{\hbar}{2} \qquad \langle \uparrow_2 | \underset{\sim}{S}_{2z} | \uparrow_2 \rangle = \frac{\hbar}{2}$$

so $\langle \uparrow_1 \uparrow_2 | \vec{\underset{\sim}{S}}_1 \cdot \vec{\underset{\sim}{S}}_2 | \uparrow_1 \uparrow_2 \rangle = +\frac{\hbar^2}{4}$

<u>also</u> $\langle \downarrow_1 \downarrow_2 | \vec{\underset{\sim}{S}}_1 \cdot \vec{S}_2 | \downarrow_1 \downarrow_2 \rangle = +\frac{\hbar^2}{4}$ $\quad = \left(-\frac{\hbar}{2}\right)^2$

claim: $\langle \uparrow_1 \uparrow_2 | \vec{\underset{\sim}{S}}_1 \cdot \vec{\underset{\sim}{S}}_2 | \text{other } 3 \rangle = 0$

$\uparrow_1 \downarrow_2, \ \downarrow_1 \uparrow_2, \ \downarrow_1 \downarrow_2$

first two: $\langle \uparrow_1 | \underset{\sim}{S}_{1x}, \underset{\sim}{S}_{1y} | \uparrow_1 \rangle = 0$ or $\langle \uparrow_2 | \underset{\sim}{S}_{2x}, \underset{\sim}{S}_{2y} | \uparrow_2 \rangle = 0$

and $\langle \uparrow_2 | \underset{\sim}{S}_{2z} | \downarrow_2 \rangle = 0$ and $\langle \uparrow_1 | \underset{\sim}{S}_{1z} | \downarrow_1 \rangle = 0$

$$\langle \uparrow_1 \uparrow_2 | \vec{\underset{\sim}{S}}_1 \cdot \vec{\underset{\sim}{S}}_2 | \downarrow_1 \downarrow_2 \rangle = \langle \uparrow_1 | \underset{\sim}{S}_{1x} | \downarrow_1 \rangle \langle \uparrow_2 | \underset{\sim}{S}_{2x} | \downarrow \rangle + \langle \uparrow_1 | \underset{\sim}{S}_{1y} | \downarrow \rangle \langle \uparrow_2 | \underset{\sim}{S}_{2y} | \downarrow \rangle$$

$$\langle\uparrow|\underset{\sim}{S}_x|\downarrow\rangle = \frac{\hbar}{2}(1\ 0)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{\hbar}{2}(1\ 0)\begin{pmatrix}0\\1\end{pmatrix} = \frac{\hbar}{2}$$

$$\langle\uparrow|\underset{\sim}{S}_y|\downarrow\rangle = \frac{\hbar}{2}(1\ 0)\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{\hbar}{2}(1\ 0)\begin{pmatrix}-i\\0\end{pmatrix} = \frac{\hbar}{2}\cdot -i$$

so $\langle\uparrow_1\uparrow_2|\underset{\sim}{S}_1\underset{\sim}{S}_2|\downarrow_1\downarrow_2\rangle = \left(\frac{\hbar}{2}\right)^2(1^2+(-i)^2) = 0\,!$

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similarly, $\langle\downarrow_1\downarrow_2|\underset{\sim}{S}_1\underset{\sim}{S}_2|$ other 3 $\rangle = 0$

$\uparrow_1\downarrow_2,\ \downarrow_1\uparrow_2,\ \uparrow_1\uparrow_2$

matrix:

	$\vert\uparrow_1\uparrow_2\rangle$	$\vert\uparrow_1\downarrow_2\rangle$	$\vert\downarrow_1\uparrow_2\rangle$	$\vert\downarrow_1\downarrow_2\rangle$
$\langle\uparrow_1\uparrow_2\vert$	$\hbar^2/4$	0	0	0
$\langle\uparrow_1\downarrow_2\vert$	0 hermitian			0 herm.
$\langle\downarrow_1\uparrow_2\vert$	0 herm.			0 hermitian
$\langle\downarrow_1\downarrow_2\vert$	0	0	0	$\hbar^2/4$

$$\langle\uparrow_1\downarrow_2|\vec{\underset{\sim}{S}}_1\cdot\vec{\underset{\sim}{S}}_2|\uparrow_1\downarrow_2\rangle = \langle\uparrow_1\downarrow_2|\underset{\sim}{S}_{1z}\cdot\underset{\sim}{S}_{2z}|\uparrow_1\downarrow_2\rangle = -\hbar^2/4$$
$$= \langle\downarrow_1\uparrow_2|\underset{\sim}{S}_{1z}\cdot\underset{\sim}{S}_{2z}|\downarrow_1\uparrow_2\rangle = -\hbar^2/4$$

($\underset{\sim}{S}_x$ + $\underset{\sim}{S}_y$ terms vanish).

<u>leaving</u>:

$$\langle\uparrow_1\downarrow_2|\vec{\underset{\sim}{S}}_1\cdot\vec{\underset{\sim}{S}}_2|\downarrow_1\uparrow_2\rangle = \langle\uparrow_1|\underset{\sim}{S}_{1x}|\downarrow_1\rangle\langle\downarrow_2|\underset{\sim}{S}_{2x}|\uparrow_2\rangle + \langle\uparrow_1|\underset{\sim}{S}_{1y}|\downarrow_1\rangle\langle\downarrow_2|\underset{\sim}{S}_{2y}|\uparrow_2\rangle$$

(z terms vanish)

$$\langle\uparrow_1|\underset{\sim}{S}_{1x}|\downarrow_1\rangle = \frac{\hbar}{2}(1\ 0)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{\hbar}{2}$$

$$\langle\downarrow_2|\underset{\sim}{S}_{2x}|\uparrow_2\rangle = \frac{\hbar}{2}(0\ 1)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}$$

$$\langle\uparrow_1|\underset{\sim}{S}_{1y}|\downarrow_1\rangle = \frac{\hbar}{2}(1\ 0)\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{\hbar}{2}(-i)$$

$$\langle\downarrow_2|\underset{\sim}{S}_{2y}|\uparrow_2\rangle = \frac{\hbar}{2}(0\ 1)\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}(+i)$$

$$\langle \uparrow_1 \downarrow_2 | \underset{\sim}{\vec{S}}_1 \cdot \underset{\sim}{\vec{S}}_2 | \downarrow_1 \uparrow_2 \rangle = \left(\frac{\hbar}{2}\right)\left(\frac{\hbar}{2}\right) + \frac{\hbar}{2}(-i) \times \frac{\hbar}{2} \times (i) = 2 \times \frac{\hbar^2}{4}$$

since $\underset{\sim}{\vec{S}}_1 \cdot \underset{\sim}{\vec{S}}_2$ is hermitian, $= \langle \downarrow_1 \uparrow_2 | \underset{\sim}{\vec{S}}_1 \cdot \underset{\sim}{\vec{S}}_2 | \uparrow_1 \downarrow_2 \rangle$

$$\underset{\sim}{\vec{S}}_1 \cdot \underset{\sim}{\vec{S}}_2 = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

if $\underset{\sim}{H} \propto \underset{\sim}{\vec{S}}_1 \cdot \underset{\sim}{\vec{S}}_2$ these terms change $\uparrow\downarrow$ into $\downarrow\uparrow$ + vice versa

recall $(\underset{\sim}{\vec{S}}_1 + \underset{\sim}{\vec{S}}_2)^2 = \underbrace{\underset{\sim}{\vec{S}}_1^2 + \underset{\sim}{\vec{S}}_2^2}_{} + 2 \underset{\sim}{\vec{S}}_1 \cdot \underset{\sim}{\vec{S}}_2$

each of these is <u>diagonal</u>

eigenvalue $2 \times \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 = \frac{3}{2}\hbar^2$

$$(\underset{\sim}{\vec{S}}_1 + \underset{\sim}{\vec{S}}_2)^2 = \frac{3}{2}\hbar^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \frac{1}{2}\hbar^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \hbar^2 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

The big point then is that $\underset{\sim}{\vec{S}}^2 = (\underset{\sim}{\vec{S}}_1 + \underset{\sim}{\vec{S}}_2)^2$ is <u>not</u> diagonal when "imaged" in the basis of product states. The little 2×2 in the middle has very simple eigenvectors:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2 \times \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{e.v.} = 2\hbar^2$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \times \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{e.v.} = 0\hbar^2$$

To then enumerate the eigenstates:

State	e.v. $\underset{\sim}{S}^2$	e.v. $\underset{\sim}{S}_z = \underset{\sim}{S}_{1z} + \underset{\sim}{S}_{2z}$
$\lvert\uparrow_1\uparrow_2\rangle$	$2\hbar^2 = 1(1+1)\hbar^2$ $S=+1$	$+\frac{\hbar}{2}+\frac{\hbar}{2} = \hbar$
$\frac{1}{\sqrt{2}}(\lvert\uparrow_1\downarrow_2\rangle + \lvert\downarrow_1\uparrow_2\rangle)$	$2\hbar^2 = 1(1+1)\hbar^2$ $S=+1$	$+\frac{\hbar}{2}-\frac{\hbar}{2}$ or - that $= 0\,\hbar$
$\frac{1}{\sqrt{2}}(\lvert\uparrow_1\downarrow_2\rangle - \lvert\downarrow_1\uparrow_2\rangle)$	$0\hbar^2 = 0(0+0)\hbar^2$ $S=0$	$0\cdot\hbar$
$\lvert\downarrow_1\downarrow_2\rangle$	$2\hbar^2 = 1(1+1)\hbar^2$ $S=+1$	$-\frac{\hbar}{2}-\frac{\hbar}{2} = -\hbar$

had $\underline{4}$ product states (2×2)

diagonalizing redistributed into 3 total spin-1 states and 1 spin-0 state.

Sometimes written as:

$$\frac{1}{2}\otimes\frac{1}{2} = \underline{1}\oplus 0$$

$$\underset{\text{states}}{2}\times\underset{\text{states}}{2} = \underset{\text{states}}{3} + \underset{\text{states}}{1}$$

<u>note</u>: "Exchange" operator swaps 1 + 2 labels

$\underset{\sim}{E}\lvert\uparrow\uparrow\rangle = \lvert\uparrow\uparrow\rangle$ eigenstate, e.v. = 1

$\underset{\sim}{E}\left[\frac{1}{\sqrt{2}}(\lvert\uparrow\downarrow\rangle+\lvert\downarrow\uparrow\rangle)\right] = \frac{1}{\sqrt{2}}(\lvert\downarrow\uparrow\rangle+\lvert\uparrow\downarrow\rangle)$ eigenstate, e.v = +1

$\underset{\sim}{E}\left[\frac{1}{\sqrt{2}}(\lvert\uparrow\downarrow\rangle-\lvert\downarrow\uparrow\rangle)\right] = \frac{1}{\sqrt{2}}(\lvert\downarrow\uparrow\rangle-\lvert\uparrow\downarrow\rangle) = -1\,\frac{1}{\sqrt{2}}(\lvert\uparrow\downarrow\rangle-\lvert\downarrow\uparrow\rangle)$ e.v. = -1

$\underset{\sim}{E}\lvert\downarrow\downarrow\rangle = \lvert\downarrow\downarrow\rangle$ e.v. = 1

e.v. = +1 ⇒ "Symmetric" e.v. = -1 "Antisymmetric"

General Case

j_1 and j_2 → form eigenstates of $(\underset{\sim}{J}_1+\underset{\sim}{J}_2)^2$

⇒ method employed for $j_1=1/2$ $j_2=1/2$ cumbersome for higher spin values

⇒ $\underset{\sim}{J}_{1z}+\underset{\sim}{J}_{2z}$ is "easy"

⇒ $(\vec{\underset{\sim}{J}}_1+\vec{\underset{\sim}{J}}_2)^2$ "hard"...

eigenvalues $J(J+1)\hbar^2$

but what values can J take?

"intuition" biggest j_1+j_2

smallest j_1-j_2 (assuming $j_1>j_2$ no loss of generality)

Question...

does $(2j_1+1)\cdot(2j_2+1) \overset{?}{=} \sum_{j=j_1-j_2}^{j_1+j_2}(2j+1)$

$j_1 \otimes j_2$

$$\sum_{j=j_1-j_2}^{j_1+j_2}(2j+1) = \sum_{j=0}^{j_1+j_2}(2j+1) - \sum_{j=0}^{j_1-j_2-1}(2j+1)$$

$$= \cancel{2}\frac{(j_1+j_2)(j_1+j_2+1)}{\cancel{2}} + (j_1+j_2+1)$$

$$-\cancel{2}\frac{(j_1-j_2-1)(j_1-j_2)}{\cancel{2}} - j_1+j_2$$

$$= \cancel{j_1^2}+2j_1j_2+\cancel{j_2^2}+j_1+j_2+j_1+j_2+1-\cancel{j_1^2}+2j_1j_2-\cancel{j_2^2}+\cancel{j_1}-\cancel{j_2}-\cancel{j_1}+\cancel{j_2}$$

$$= 4j_1j_2+2j_1+2j_2+1 = (2j_1+1)(2j_2+1)$$