STAEDTLER® No. 937 811E Engineer's Computation Pad

$$\underset{\sim}{M} = m_0 \underset{\sim}{\sigma}_0 + \underbrace{m_1 \underset{\sim}{\sigma}_1 + m_2 \underset{\sim}{\sigma}_2 + m_3 \underset{\sim}{\sigma}_3}_{}$$

$\uparrow$ same in all bases

$$= \sqrt{m_1^2+m_2^2+m_3^2} \times \left\{ \sin\theta\cos\phi\, \underset{\sim}{\sigma}_1 + \sin\theta\sin\phi\, \underset{\sim}{\sigma}_2 + \cos\theta\, \underset{\sim}{\sigma}_3 \right\}$$

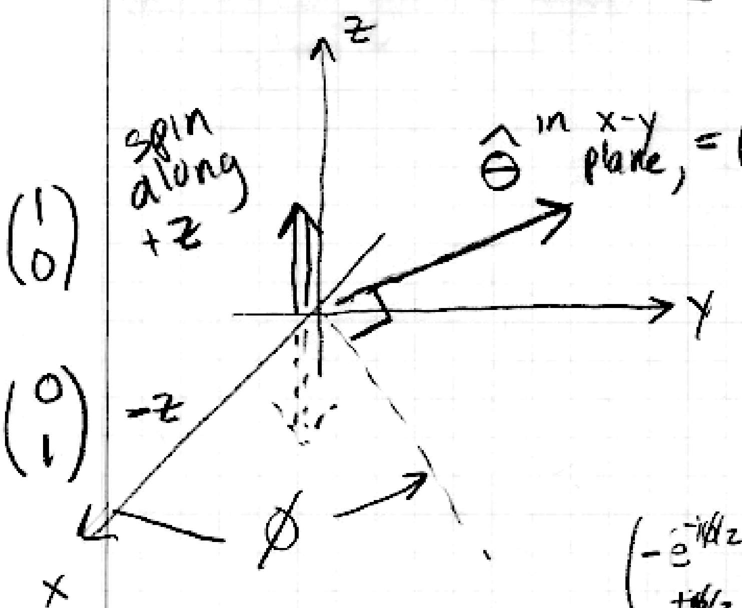
where: $\cos\theta \equiv \dfrac{m_3}{\sqrt{m_1^2+m_2^2+m_3^2}}$ $\quad \tan\phi = \dfrac{m_2}{m_1}$

but, eigenvalues of $\sin\theta\cos\phi\, \underset{\sim}{\sigma}_1 + \sin\theta\sin\phi\, \underset{\sim}{\sigma}_2 + \cos\theta\, \underset{\sim}{\sigma}_3$ are... ± 1

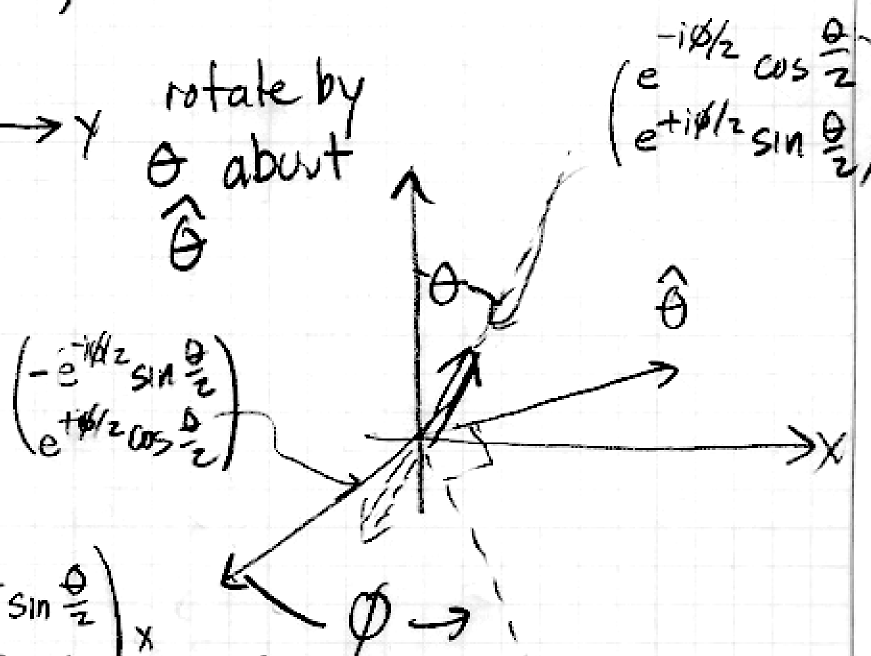
eigenvectors are... $\begin{pmatrix} e^{-i\phi/2}\cos\frac{\theta}{2} \\ e^{+i\phi/2}\sin\frac{\theta}{2} \end{pmatrix}, \begin{pmatrix} -e^{-i\phi/2}\sin\frac{\theta}{2} \\ e^{+i\phi/2}\cos\frac{\theta}{2} \end{pmatrix}$

overall eigenvalues are then: $= m_0 \pm \sqrt{m_1^2+m_2^2+m_3^2}$

Rotation connection!



rotate by θ about $\hat{\theta}$



$$\text{Rotated} = \underbrace{\begin{pmatrix} e^{-i\phi/2}\cos\frac{\theta}{2} & -e^{-i\phi/2}\sin\frac{\theta}{2} \\ e^{+i\phi/2}\sin\frac{\theta}{2} & e^{i\phi/2}\cos\frac{\theta}{2} \end{pmatrix}}_{\underset{\sim}{U}(R(\vec{\theta}))} \times \text{Unrotated}.$$

$\vec{\theta} = \theta \times \hat{\theta}$

$$\underset{\sim}{U}(R(\vec{\theta})) = \lim_{N\to\infty}\left(1 - i\frac{\vec{\theta}\cdot\vec{\underset{\sim}{S}}}{N\hbar}\right)^N = \exp\left(-i\frac{\vec{\theta}\cdot\vec{\underset{\sim}{\sigma}}}{2}\right)$$

see p. 290 $\vec{\theta} \to$ like $\vec{a}$

$\vec{\underset{\sim}{S}} \to$ like $\vec{\underset{\sim}{R}}$

$$\exp\left(-i\frac{\vec{\theta}\cdot\vec{\underset{\sim}{\sigma}}}{2}\right) = \underset{\sim}{\mathbb{1}} + \left(\frac{-i\theta}{2}\right)\hat{\theta}\cdot\vec{\underset{\sim}{\sigma}} + \frac{1}{2!}\left(\frac{-i\theta}{2}\right)^2(\hat{\theta}\cdot\vec{\underset{\sim}{\sigma}})^2 + \frac{1}{3!}\left(\frac{-i\theta}{2}\right)^3 \times(\hat{\theta}\cdot\vec{\underset{\sim}{\sigma}})^3$$

but.. $(\hat{\theta}\cdot\vec{\underset{\sim}{\sigma}})^2 = \underset{\sim}{\mathbb{1}}$ ... series collapses to 2 terms

$$\exp\left(-i\frac{\vec{\theta}\cdot\vec{\sigma}}{2}\right) = \underset{\sim}{\mathbb{1}}\left(1 - \frac{1}{2!}\left(\frac{\theta}{2}\right)^2 + \frac{1}{4!}\left(\frac{\theta}{2}\right)^4 \cdots\right)$$
$$- i(\hat{\theta}\cdot\vec{\underset{\sim}{\sigma}})\left(\frac{\theta}{2} - \frac{1}{3!}\left(\frac{\theta}{2}\right)^3 + \cdots\right)$$
$$= \cos\left(\frac{\theta}{2}\right)\cdot\underset{\sim}{\mathbb{1}} - i\sin\frac{\theta}{2}(\hat{\theta}\cdot\vec{\underset{\sim}{\sigma}})$$

$$\hat{\theta}\cdot\vec{\underset{\sim}{\sigma}} = -\sin\phi\,\underset{\sim}{\sigma}_1 + \cos\phi\,\underset{\sim}{\sigma}_2$$
$$= \begin{pmatrix} 0 & -\sin\phi - i\cos\phi \\ -\sin\phi + i\cos\phi & 0 \end{pmatrix} = \begin{pmatrix} 0 & -ie^{-i\phi} \\ ie^{i\phi} & 0 \end{pmatrix}$$

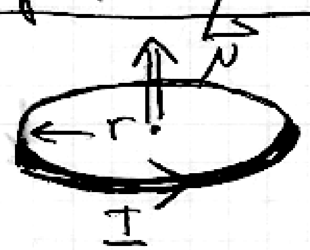
$$\cos\frac{\theta}{2}\cdot\underset{\sim}{\mathbb{1}} - i\sin\frac{\theta}{2}(\hat{\theta}\cdot\vec{\underset{\sim}{\sigma}}) = \begin{pmatrix} \cos\frac{\theta}{2} & 0 \\ 0 & \cos\frac{\theta}{2} \end{pmatrix} + \begin{pmatrix} 0 & -\sin\frac{\theta}{2}e^{-i\phi} \\ \sin\frac{\theta}{2}e^{i\phi} & 0 \end{pmatrix}$$

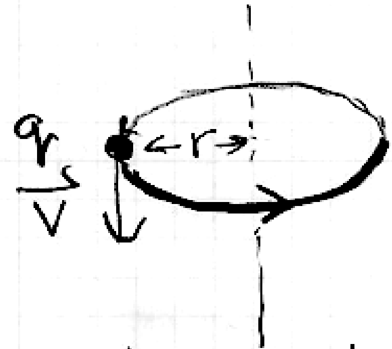
$$\exp\left(-i\frac{\vec{\theta}\cdot\vec{\sigma}}{2}\right) = \begin{pmatrix} \cos\frac{\theta}{2} & -e^{-i\phi}\sin\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

overall phase is meaningless   overall phase is meaningless

$$\Rightarrow \text{same as} \begin{pmatrix} e^{-i\phi/2}\cos\frac{\theta}{2} & -e^{-i\phi/2}\sin\frac{\theta}{2} \\ e^{i\phi/2}\sin\frac{\theta}{2} & e^{i\phi/2}\cos\frac{\theta}{2} \end{pmatrix}$$

Spin Dynamics





$$|\vec{\mu}| = \frac{I \cdot A}{c} = \frac{\pi r^2 I}{c}$$

direction: $\perp$ to loop, right hand rule.

in time Δt, charge travels $v\Delta t$, passes by $\frac{v\Delta t}{2\pi r}$ times.

$$I = \frac{Q}{\Delta t} = \frac{1}{\Delta t}\frac{v\Delta t}{2\pi r} \times q$$

$$|\vec{\mu}| = \frac{\pi r^2}{c} \times \frac{v}{2\pi r} \times q = \frac{q}{2mc}\underbrace{(mvr)}_{\text{orbital angular momentum } |\vec{L}|}$$

comparing directions:

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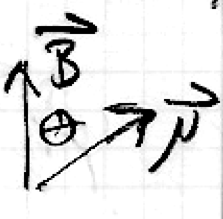
$$\vec{\mu} = \frac{q}{2mc}\vec{L}$$

(orbital angular momentum).

Origin of spin not well understood, so a fudge factor, g, is introduced for fundamental particles like the electron, which has $q = -e$.

Electron: $\vec{\mu} = -\frac{ge}{2mc}\vec{S}$ $\quad g \simeq 2$ • experiment • relativity.

Torque: $\vec{T} = \vec{\mu} \times \vec{B} \propto \sin\theta$

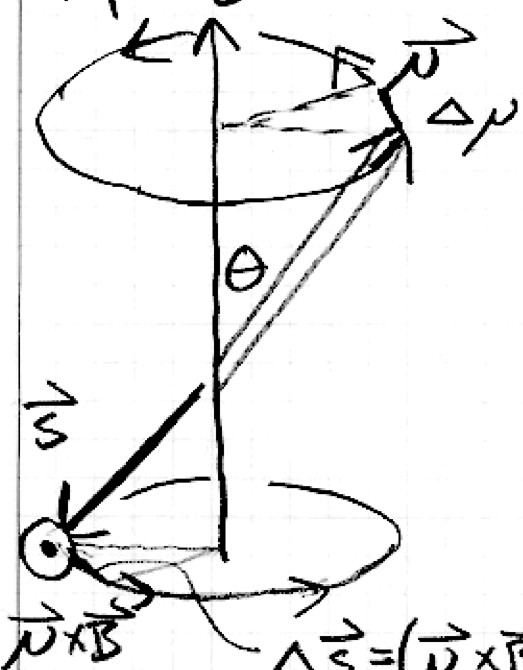


wants to line up $\vec{\mu}$ and $\vec{B}$.

this is prop to $U'(\theta)$, where $U(\theta)$ is the potential energy associated with the torque...

$$U(\theta) \propto -\cos\theta$$
$$= -|\vec{\mu}||\vec{B}|\cos\theta$$
$$= -\vec{\mu}\cdot\vec{B}$$

An electron at rest in a $\vec{B}$ field <u>precesses</u> due to the torque, just like a top:



$$\Delta\phi = \frac{\Delta\mu}{\mu\sin\theta} = \frac{\Delta s}{s\sin\theta}$$

$$\Delta\phi = \frac{geB}{2mc}\Delta t \quad \left(\text{independent of } \theta\right)$$

$$\omega_0 = \frac{\Delta\phi}{\Delta t} = \frac{geB}{2mc}$$

$$\Delta\vec{s} = (\vec{\mu}\times\vec{B})\Delta t$$

$$|\Delta\vec{s}| = \frac{ge}{2mc} sB\sin\theta\,\Delta t$$

more generally,

$$\gamma \equiv \frac{q}{2mc}$$

and

$$\vec{\omega}_0 = -g\gamma\vec{B}$$

is the precession frequency.

this was "classical".

QM: $$i\hbar\frac{\partial}{\partial t}\psi = \underset{\sim}{H}\psi$$

$$\underset{\sim}{U}(t) = \lim_{N\to\infty}\left(\underset{\sim}{\mathbb{1}} - \frac{i\underset{\sim}{H}t}{\hbar N}\right)^N = e^{-\frac{i\underset{\sim}{H}t}{\hbar}}$$

$$\underset{\sim}{H} = -\underset{\sim}{\vec{\mu}} \cdot \vec{B} = +\frac{ge}{2mc}\underset{\sim}{\vec{S}} \cdot \vec{B} \quad \text{(electron)}$$

but $\vec{B} = B\hat{n}$ $\quad \hat{n} \rightarrow$ direction of $\vec{B}$

$$\underset{\sim}{H} = \frac{geB}{2mc}\hat{n} \cdot \underset{\sim}{\vec{S}} = \frac{geB\hbar}{4mc}\hat{n} \cdot \underset{\sim}{\vec{\sigma}}$$

$$\underset{\sim}{U}(t) = e^{-\frac{igeBt}{4mc}\hat{n} \cdot \vec{\sigma}}$$

note: $\omega_0 = g\gamma B = \frac{ge}{2mc}B$

$$\underset{\sim}{U}(t) = e^{-\frac{i}{2}(\omega_0 t)\hat{n} \cdot \underset{\sim}{\vec{\sigma}}}$$

$\rightarrow \underset{\sim}{U}(t)$ causes a <u>rotation</u> by angle $\omega_0 t$ <u>about</u> the direction of the $\vec{B}$ field. This is exactly the same as precession!