$$\lambda = -\frac{\hbar}{2} \quad \begin{pmatrix} \hbar/2 & \hbar/2 \\ \hbar/2 & \hbar/2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$$

$$\alpha = -\beta$$

eigenstate $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\underset{\sim}{S}_y$!

$$\underset{\sim}{S}_x \underset{\sim}{S}_z - \underset{\sim}{S}_z \underset{\sim}{S}_x = \underset{\sim}{S}_1 \underset{\sim}{S}_3 - \underset{\sim}{S}_3 \underset{\sim}{S}_1 = i\hbar\, \varepsilon_{132} \underset{\sim}{S}_2$$

$$= -i\hbar \underset{\sim}{S}_2$$

$$\underset{\sim}{S}_y = \underset{\sim}{S}_2 = \frac{1}{-i\hbar}\left(\frac{\hbar}{2}\right)^2 \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right]$$

$$= \;'' \;\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right]$$

$$= \frac{i\hbar}{4}\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\uparrow$$

$$\underset{\sim}{\sigma}_y \text{ or } \underset{\sim}{\sigma}_2$$

<u>Homework</u>: expectation value of $\underset{\sim}{S}_y$; eigenvalues, states of $\underset{\sim}{S}_y$.

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eigenvector connection:

A $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is $\begin{pmatrix} e^{-i\phi/2}\cos\frac{\theta}{2} \\ e^{i\phi/2}\sin\frac{\theta}{2} \end{pmatrix}$ with $\phi = 0$, $\theta = 0$ "up"

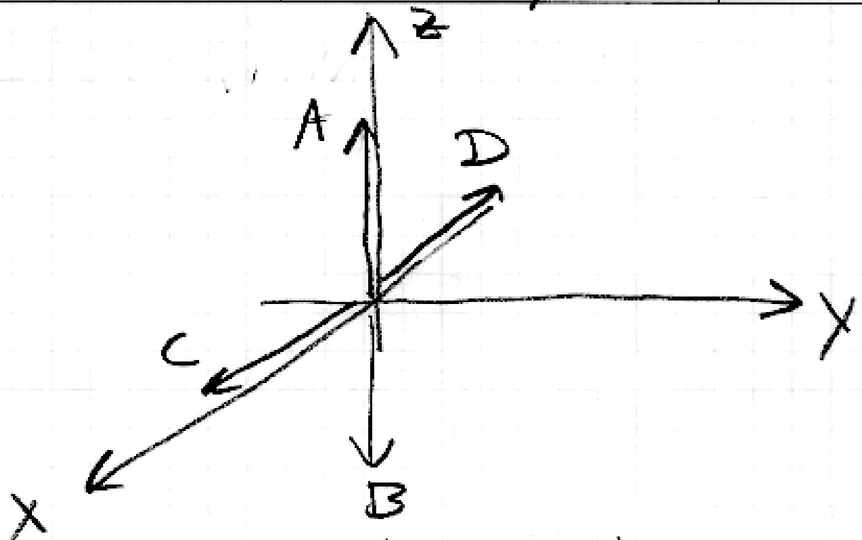
$\underset{\sim}{S}_z$

B $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is $\begin{pmatrix} e^{-i\phi/2}\cos\frac{\theta}{2} \\ e^{i\phi/2}\sin\frac{\theta}{2} \end{pmatrix}$ with $\phi = 0$, $\theta = \pi$ "down"

C $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is " with $\phi = 0$, $\theta = \pi/2$ "+x"

$\underset{\sim}{S}_x$

D $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\Big(\quad \Big)$ with $\phi = \pi$, $\theta = \frac{\pi}{2}$ $\rightarrow \frac{-i}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ "-x"



idea: $\begin{pmatrix} e^{-i\phi/2} \cos\frac{\theta}{2} \\ e^{i\phi/2} \sin\frac{\theta}{2} \end{pmatrix}$ is the eigenvector of a spin operator along the direction θ, ϕ (w e.v. $+\hbar/2$)

$$\underset{\sim}{S}_n = \cos\theta\, \underset{\sim}{S}_z + \sin\theta\cos\phi\, \underset{\sim}{S}_x + \sin\theta\sin\phi\, \underset{\sim}{S}_y$$

$$\hat{n} = \cos\theta\, \hat{z} + \sin\theta\cos\phi\, \hat{x} + \sin\theta\sin\phi\, \hat{y}$$

$$\vec{\underset{\sim}{S}} \equiv \underset{\sim}{S}_z \hat{z} + \underset{\sim}{S}_x \hat{x} + \underset{\sim}{S}_y \hat{y}$$

$$\underset{\sim}{S}_n \equiv \hat{n} \cdot \vec{\underset{\sim}{S}} \qquad \text{(p. 380)}$$

eigenvector with eigenvalue $= -\hbar/2$?

$$|n-\rangle = \begin{bmatrix} -e^{-i\phi/2} \sin\theta/2 \\ e^{i\phi/2} \cos\theta/2 \end{bmatrix} \quad \text{e.v.} -\hbar/2$$

label

$$|n+\rangle = \begin{bmatrix} e^{-i\phi/2} \cos\theta/2 \\ e^{i\phi/2} \sin\theta/2 \end{bmatrix} \quad \text{e.v.} +\hbar/2$$

Pauli Matrices

$$\underset{\sim}{S}_1 = \underset{\sim}{S}_x = \frac{\hbar}{2}\underset{\sim}{\sigma}_1 = \frac{\hbar}{2}\underset{\sim}{\sigma}_x \; ; \quad \underset{\sim}{\sigma}_1 = \underset{\sim}{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\underset{\sim}{S}_2 = \underset{\sim}{S}_y = \frac{\hbar}{2}\underset{\sim}{\sigma}_2 = \frac{\hbar}{2}\underset{\sim}{\sigma}_y \; ; \quad \underset{\sim}{\sigma}_2 = \underset{\sim}{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\underset{\sim}{S}_3 = \underset{\sim}{S}_z = \frac{\hbar}{2}\underset{\sim}{\sigma}_3 = \frac{\hbar}{2}\underset{\sim}{\sigma}_z \; ; \quad \underset{\sim}{\sigma}_3 = \underset{\sim}{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\boxed{\text{simply memorize}}$

Properties

(1) They **anti**commute

$$\underset{\sim}{\sigma}_i \underset{\sim}{\sigma}_j + \underset{\sim}{\sigma}_j \underset{\sim}{\sigma}_i \equiv [\underset{\sim}{\sigma}_i, \underset{\sim}{\sigma}_j]_+ \equiv \{\underset{\sim}{\sigma}_i, \underset{\sim}{\sigma}_j\}$$
$$= 2\delta_{ij} \times \underset{\sim}{\mathbb{1}}$$

(14.3.32 p. 381)
14.3.35
14.3.38 p. 382

proof: simply grind through

$$\underset{\sim}{\sigma}_1 \underset{\sim}{\sigma}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \; ; \quad \underset{\sim}{\sigma}_2 \underset{\sim}{\sigma}_1 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$\underset{\sim}{\sigma}_1 \underset{\sim}{\sigma}_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \; ; \quad \underset{\sim}{\sigma}_3 \underset{\sim}{\sigma}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\underset{\sim}{\sigma}_2 \underset{\sim}{\sigma}_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \; ; \quad \underset{\sim}{\sigma}_3 \underset{\sim}{\sigma}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

$$\underset{\sim}{\sigma}_1^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ; \; \underset{\sim}{\sigma}_2^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ; \; \underset{\sim}{\sigma}_3^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- not generally true for representations of higher angular momenta.

(2) $\underset{\sim}{\sigma}_i \underset{\sim}{\sigma}_j = i\,\varepsilon_{ijk}\underset{\sim}{\sigma}_k + \delta_{ij} \times \underset{\sim}{\mathbb{1}}$ (see above)

$$\underset{\sim}{\sigma}_1 \underset{\sim}{\sigma}_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\underset{\sim}{\sigma}_3 \quad \underset{\sim}{\sigma}_1 \underset{\sim}{\sigma}_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -i\underset{\sim}{\sigma}_2$$

$$\underset{\sim}{\sigma}_2 \underset{\sim}{\sigma}_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\underset{\sim}{\sigma}_1$$

(14.3.33 p. 381)
14.3.35

(3) $\mathrm{Tr}(\underset{\sim}{\sigma}_i) = 0$ just look at them!

$\mathrm{Tr}(\underset{\sim}{\sigma}_1) = \mathrm{Tr}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0 + 0 = 0$ $\mathrm{Tr}(\underset{\sim}{\sigma}_2) = \mathrm{Tr}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = 0 + 0 = 0$

$\mathrm{Tr}(\underset{\sim}{\sigma}_3) = \mathrm{Tr}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 1 - 1 = 0$ (14.3.34 p. 381)

(4) $(\hat{n} \cdot \underset{\sim}{\vec{\sigma}})^2 = \underset{\sim}{\mathbb{1}}$ $\hat{n} = (n_x, n_y, n_z)$, $n_x^2 + n_y^2 + n_z^2 = 1$

$(n_x \underset{\sim}{\sigma}_x + n_y \underset{\sim}{\sigma}_y + n_z \underset{\sim}{\sigma}_z)^2$

$= n_x^2 \underbrace{\underset{\sim}{\sigma}_x^2}_{\mathbb{1}} + n_y^2 \underbrace{\underset{\sim}{\sigma}_y^2}_{\mathbb{1}} + n_z^2 \underbrace{\underset{\sim}{\sigma}_z^2}_{\mathbb{1}}$

$+ n_x n_y \underbrace{(\underset{\sim}{\sigma}_x \underset{\sim}{\sigma}_y + \underset{\sim}{\sigma}_y \underset{\sim}{\sigma}_x)}_{0} + n_x n_z \underbrace{(\underset{\sim}{\sigma}_x \underset{\sim}{\sigma}_z + \underset{\sim}{\sigma}_z \underset{\sim}{\sigma}_x)}_{0}$

$+ n_y n_z \underbrace{(\underset{\sim}{\sigma}_y \underset{\sim}{\sigma}_z + \underset{\sim}{\sigma}_z \underset{\sim}{\sigma}_y)}_{0}$

$= (n_x^2 + n_y^2 + n_z^2) \cdot \underset{\sim}{\mathbb{1}} = \underset{\sim}{\mathbb{1}}$.

(6) $\underbrace{(\vec{\underset{\sim}{A}} \cdot \underset{\sim}{\vec{\sigma}})(\vec{\underset{\sim}{B}} \cdot \underset{\sim}{\vec{\sigma}})}_{\text{2 powers of } \underset{\sim}{\sigma}\text{'s}} = \underbrace{(\vec{A} \cdot \vec{B})\underset{\sim}{\mathbb{1}}}_{\text{0 powers of } \underset{\sim}{\sigma}\text{'s}} + \underbrace{i(\vec{A} \times \vec{B}) \cdot \underset{\sim}{\vec{\sigma}}}_{\text{1 power of } \underset{\sim}{\sigma}}$

$(A_x \underset{\sim}{\sigma}_x + A_y \underset{\sim}{\sigma}_y + A_z \underset{\sim}{\sigma}_z)(B_x \underset{\sim}{\sigma}_x + B_y \underset{\sim}{\sigma}_y + B_z \underset{\sim}{\sigma}_z) = \ldots$

- "direct terms" $\rightarrow \vec{A} \cdot \vec{B} \cdot \underset{\sim}{\mathbb{1}}$
- "cross terms" $\rightarrow i(\vec{A} \times \vec{B}) \cdot \underset{\sim}{\vec{\sigma}}$.

(will be homework).

(7) $\mathrm{Tr}(\underset{\sim}{\sigma}_i \underset{\sim}{\sigma}_j) = 2$ when $i = j$

$= -\mathrm{Tr}(\underset{\sim}{\sigma}_j \underset{\sim}{\sigma}_i)$ when $i \neq j$ (see (1))

$= \mathrm{Tr}(\underset{\sim}{\sigma}_j \underset{\sim}{\sigma}_i)$ property of trace

$\mathrm{Tr}(\underset{\sim}{A}\underset{\sim}{B}) = \sum_{ij} A_{ij} B_{ji} = \sum_{ij} B_{ji} A_{ij} = \mathrm{Tr}(\underset{\sim}{B}\underset{\sim}{A})$

$\therefore \mathrm{Tr}(\underset{\sim}{\sigma}_i \underset{\sim}{\sigma}_j) = 0$ when $i \neq j$

$\mathrm{Tr}(\underset{\sim}{\sigma}_i \underset{\sim}{\sigma}_j) = 2\delta_{ij}$

add to these 3 $\underset{\sim}{\sigma}_0 \equiv \underset{\sim}{\mathbb{1}}$; make greek index α or β range from 0 to 3

Note $\underset{\sim}{\sigma}_0$ is "special", <u>representation</u> <u>independent</u>.

$$\mathrm{Tr}(\underset{\sim}{\sigma}_\alpha \underset{\sim}{\sigma}_\beta) = 2\delta_{\alpha\beta} \qquad \alpha,\beta \to 0,1,2,3.$$

The trace functions like an inner product except for a space spanned not by vectors, but by 2×2 matrices. Pretty much:

$$\underset{\sim}{M} = \sum_{\alpha=0}^{4} m_\alpha \underset{\sim}{\sigma}_\alpha$$

arbitrary 2×2 matrix

expansion over the 4 basis matrices.

$$\mathrm{Tr}(\underset{\sim}{M}\underset{\sim}{\sigma}_\beta) = \mathrm{Tr}\left(\sum_{\alpha=0}^{4} m_\alpha \underset{\sim}{\sigma}_\alpha \underset{\sim}{\sigma}_\beta\right) = \sum_{\alpha=0}^{4} m_\alpha \mathrm{Tr}(\underset{\sim}{\sigma}_\alpha \underset{\sim}{\sigma}_\beta)$$

$$= \sum_{\alpha=0}^{4} m_\alpha 2\delta_{\alpha\beta} = 2m_\beta$$

$$m_\beta = \frac{1}{2}\mathrm{Tr}(\underset{\sim}{M}\underset{\sim}{\sigma}_\beta)$$

when $\underset{\sim}{M}$ is Hermitian, then all the m_β are real.

$\underset{\sim}{M}^\dagger$ ($= \underset{\sim}{M}$ means $\underset{\sim}{M}$ is Hermitian)

$$= \sum_{\alpha=0}^{4} m_\alpha^* \underset{\sim}{\sigma}_\alpha^\dagger \qquad \text{but } \underset{\sim}{\sigma}_\alpha^\dagger = \underset{\sim}{\sigma}_\alpha$$

$$= \sum_{\alpha=0}^{4} m_\alpha^* \sigma_\alpha = \sum_{\alpha=0}^{4} m_\alpha \underset{\sim}{\sigma}_\alpha \Rightarrow m_\alpha = m_\alpha^*$$

but now this allows simple and efficient diagonalization of $\underset{\sim}{M}$ (when Hermitian)

How?