## Spin

Suppose we take a particle and prepare it in a wave-function known to have no orbital angular momentum.

Measure $\underset{\sim}{L}_x$ → get 0 every time ($\overset{0}{\text{eigen value}}$)

x-component of <u>orbital</u> angular momentum.

BTW: $\underset{\sim}{L}_x = \underset{\sim}{y}\, p_z - \underset{\sim}{z}\, p_y$

Same for $\underset{\sim}{L}_y, \underset{\sim}{L}_z$ → measure 0 every time.

Now suppose we devise a method to measure the total angular momentum. One realization of this is to use <u>precession</u> of the particle.

Turns out we <u>don't</u> see: $\underset{\sim}{J}_x$ → e.v. 0 every time

(p. 373)

$\underset{\sim}{J}_y, \underset{\sim}{J}_z$ → e.v. 0 every time

→ conclude, there is an <u>extra</u>, "intrinsic" angular momentum... "intrinsic" in the sense that we cannot alter it (yet!)

total:

$$\underset{\sim}{J}_x = \underset{\sim}{L}_x + \underset{\sim}{S}_x$$

$$\underset{\sim}{J}_y = \underset{\sim}{L}_y + \underset{\sim}{S}_y$$

$$\underset{\sim}{J}_z = \underset{\sim}{L}_z + \underset{\sim}{S}_z$$

"Total" or "Total J" ... "spin"

For $\underset{\sim}{J}_i$ ($\underset{\sim}{J}_1 = \underset{\sim}{J}_x$, $\underset{\sim}{J}_2 = \underset{\sim}{J}_y$, $\underset{\sim}{J}_3 = \underset{\sim}{J}_z$) to really be an angular momentum,

$$[\underset{\sim}{J}_i, \underset{\sim}{J}_j] = i\hbar \sum_k \varepsilon_{ijk} \underset{\sim}{J}_k \quad = i\varepsilon_{ijk}\underset{\sim}{J}_k$$

(sum implied by repeated index)

imaginary number

index

$\varepsilon_{ijk} = 0$ unless all three indices distinct

(p. 375)

$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = +1$

$\varepsilon_{321} = \varepsilon_{213} = \varepsilon_{132} = -1$

must be: $[\underset{\sim}{S}_i, \underset{\sim}{S}_j] = i\hbar \sum_k \varepsilon_{ijk} \underset{\sim}{S}_k$

Categorization of eigenkets, eigenvalues, complete set of commuting observables (see pp. 324-5 of text)

$\underset{\sim}{S}^2 = \underset{\sim}{S}_x^2 + \underset{\sim}{S}_y^2 + \underset{\sim}{S}_z^2 \rightarrow$ eigenvalue $s(s+1)\hbar^2$

$s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

half integer now <u>allowed</u>; no <u>spatial</u> wave function

Degeneracy: $2s+1$

Distinguished by $\underset{\sim}{S}_z$: eigenvalues $\hbar\cdot(-s, -s+1, \ldots s-1, s)$

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First interesting case: $s = 1/2$

Dimensionality: 2 can represent everything in a 2×2 space of matrices (operators).

(p. 376) two component column vectors

$$|\psi\rangle \rightarrow \begin{bmatrix}\psi_+(\vec{x})\\ \psi_-(\vec{x})\end{bmatrix} \qquad \vec{P} \longrightarrow \frac{\hbar}{i}\begin{bmatrix}\vec{\nabla} & 0\\ 0 & \vec{\nabla}\end{bmatrix}$$

state

Nearly always we work with a state, when studying spin, that have 0 momentum.

$$\vec{P}|\psi\rangle = 0 \rightarrow \frac{\hbar}{i}\begin{bmatrix}\vec{\nabla} & 0\\ 0 & \vec{\nabla}\end{bmatrix}\begin{bmatrix}\psi_+(\vec{x})\\ \psi_-(\vec{x})\end{bmatrix} = 0$$

$\psi_+(\vec{x}), \psi_-(\vec{x})$ = constants. (as a function of $\vec{x}, \vec{y}, \vec{z}$).

This describes a particle at rest.

$|\psi\rangle \rightarrow \begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ $\quad \alpha, \beta$ = complex #'s (4 #'s)

$|\alpha|^2 + |\beta|^2 = 1$ (1 constraint)

$4-1 = 3$ #'s

hmm....

changing $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ to $e^{i\gamma}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$

with $\gamma$ = real # doesn't change physics

$3-1 = 2$ #'s relevant in $\alpha, \beta$

since $|\alpha|^2 + |\beta|^2 = 1$ can take

$\alpha = \cos\frac{\theta}{2} \quad \beta = \sin\frac{\theta}{2} \quad \theta = 1$ of 2 real #'s

since overall phase is irrelevant, the most general

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}e^{-i\phi/2}\cos\frac{\theta}{2}\\ e^{+i\phi/2}\sin\frac{\theta}{2}\end{bmatrix}$$

what is the physical meaning of this?

Need some Hermitian operators that represent observables. Since this is only a 2×2 space, it is not hard to enumerate a "basis" that "spans" the space of 2×2 Hermitian operators:

$$\underset{\sim}{I} \doteq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ expectation value:

$$\langle\psi|\underset{\sim}{I}|\psi\rangle = \langle\psi|\psi\rangle$$

$$= \begin{bmatrix}\alpha^* & \beta^*\end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix}\alpha \\ \beta\end{bmatrix}$$

"Just the normalization" $= |\alpha|^2 + |\beta|^2 = \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1$

→ no other physics

however, arbitrary $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ not $\propto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

real #'s

but, $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \frac{1}{2}(A+B)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}(A-B)\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\text{"}\underset{\sim}{\sigma}_z\text{" or } \underset{\sim}{\sigma}_3}$

<u>Physics</u> <u>Available</u>:

$$\langle\psi|\underset{\sim}{\sigma}_z|\psi\rangle = \begin{bmatrix}\alpha^* & \beta^*\end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = |\alpha|^2 - |\beta|^2$$

$$= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \cos 2\times\frac{\theta}{2} = \cos\theta$$

Like the z-component of unit vector!

hmm... why not

$$\underset{\sim}{S}_z \doteq \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar}{2}\underset{\sim}{\sigma}_z$$

→ $|\alpha|^2$ prob measure $+\frac{\hbar}{2}$, $\begin{pmatrix}1 \\ 0\end{pmatrix}$

→ $|\beta|^2$ prob measure $-\frac{\hbar}{2}$, $\begin{pmatrix}0 \\ 1\end{pmatrix}$

eigen vectors

so expectation value of $\underset{\sim}{S}_z = \dfrac{+\frac{\hbar}{2}|\alpha|^2 - \frac{\hbar}{2}|\beta|^2}{|\alpha|^2+|\beta|^2 \rightarrow 1}$

$$\langle \underset{\sim}{S}_z \rangle = \frac{\hbar}{2}\left(|\alpha|^2 - |\beta|^2\right) = \frac{\hbar}{2}\cos\theta$$

Expectation Value behaves like a component of a vector.

What about $\underset{\sim}{S}_x$, $\underset{\sim}{S}_y$ ?

Hermitian... 2x2 ... what is left ?

$\underset{\sim}{\sigma}_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ not diagonal! don't know eigenvalues (y also)

or $\underset{\sim}{\sigma}_1$

$$\langle \psi | \underset{\sim}{\sigma}_x | \psi \rangle = [\alpha^* \ \beta^*] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha^*\beta + \alpha\beta^*$$

$$= e^{i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} + e^{-i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2}$$

$$= \underbrace{\left[\frac{1}{2}\left(e^{i\phi}+e^{-i\phi}\right)\right]}_{\cos\phi} \underbrace{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}_{\sin\theta}$$

$$\langle \psi | \underset{\sim}{\sigma}_x | \psi \rangle = \sin\theta\cos\phi$$

why not... $\underset{\sim}{S}_x = \frac{\hbar}{2}\underset{\sim}{\sigma}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

not diagonal!

$$\begin{vmatrix} -\lambda & \hbar/2 \\ \hbar/2 & -\lambda \end{vmatrix} = \lambda^2 - \frac{\hbar}{2}^2 = 0, \quad \lambda = \pm \hbar/2$$

eigenstates ! $\lambda = +\frac{\hbar}{2}$ $\begin{pmatrix} -\hbar/2 & \hbar/2 \\ \hbar/2 & -\hbar/2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$

$$\alpha = \beta$$

eigenstate $= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$