3) Fall to the Center, finite/infinite # of Bound States

For attractive potentials, classical solution of radial problem with lowest energy would always be $r=0$; for $s<0$ would mean that the particle would "fall to the center".

Further, the existence of radiation means that an electron, classically, would emit radiation until it fell down on top of the positively charged nucleus.

Quantum mechanics inhibits the fall to the center via the uncertainty principle:

localization $\rightarrow$ large momentum components
$\downarrow$
high expectation value of kinetic energy. $\langle T \rangle$
$\downarrow$
total energy $(=\langle T\rangle + \langle V\rangle)$ then is raised off $-\infty$ to a finite value

Point to discuss here is: could a potential be so strong as to suck even a quantum mechanical particle down to the center?

first, $\langle T \rangle = \left\langle \frac{p^2}{2\mu} \right\rangle = \frac{1}{2\mu}\left[\underbrace{\cancel{\langle p\rangle}^2}_{0} + (\Delta p)^2\right]$

0 state at rest

but $\Delta x \Delta p \gtrsim \frac{\hbar}{2}$

Order of magnitude: $\langle T \rangle \gtrsim \frac{\hbar^2}{\mu(\Delta x)^2}$

Now suppose the particle is confined

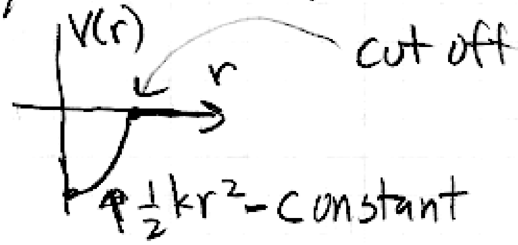
within a radius r_0 in an attractive power law potential, $-|V_0| r^s$, $s < 0$.

Then $\langle V \rangle \lesssim \dfrac{-|V_0|}{r_0^{|s|}}$

and $\langle E \rangle \cong \langle T \rangle + \langle V \rangle \lesssim \dfrac{\hbar^2}{\mu r_0^2} - \dfrac{|V_0|}{r_0^{|s|}}$

Point is: when $|s| > 2$ or $s < -2$, the potential can beat the uncertainty principle. But the Coulomb interaction, with $s = -1$, can't!

The finite/infinite # of bound states applies when $V(r) \to 0$ as $r \to 0$. If $s > 0$, you have to "cut of" the potential like



. If $s < 0$, cut off happens naturally.

Make a wavefunction that is localized within Δr, at radius r_0. There is detail that $U(r)$ is localized to Δr, to determine the energy, but $R(r) = U(r)/r$ should be held constant to keep the wavefunction alike as we vary r_0.

So we take $\Delta r \propto r_0$, $= \alpha r_0$

Then $\langle E \rangle \lesssim \dfrac{\hbar^2}{\mu \Delta r^2} - \dfrac{|V_0|}{r_0^{|s|}} = \dfrac{\hbar^2}{\mu a r_0^2} - \dfrac{|V_0|}{r_0^{|s|}}$

here, if $1/r_0^{|s|}$ falls off slower than $1/r_0^2$, eventually the negative term outpaces the positive one, and one gets a bound state with $\langle E\rangle < 0$. Since Δr can be arbitrarily large, the binding energy can be arbitrarily small. In other words, as $r\to\infty$, if $V(r)\to 0$ more slowly than $-1/r_0^2$, there will be an infinite # of bound states.

as $r\to 0$, if $V(r)\to 0$ more quickly than $1/r_0^2$, there are a finite # of bound states.

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4) Functional behavior as $r\to\infty$ when $V(r)\to 0$ (but slower than $1/r^2$) and $E<0$

This is "tunneling" situation. For very large r, $\ell(\ell+1)/r^2$ can be ignored relative to $V(r)$. Going back to the dimensioned radial equation,

$$\left\{\frac{\hbar^2}{2\mu}\left[-\frac{d^2}{dr^2}+\cancel{\frac{\ell(\ell+1)}{r^2}}\right]+V(r)\right\}U_{E\ell}(r)=-|E|U_{E\ell}(r)$$

assume $V(r)$ dominates $r\to\infty$

$$\frac{\hbar^2}{2\mu}\left(-\frac{d^2}{dr^2}+V(r)\right)U_E(r)=-|E|U_E(r)$$

since $V(r)\to 0$ as $r\to\infty$, let's try to use the solution we obtain when $V(r)=0$, and then modify it. When $V(r)=0$,

$$U_E(r) = e^{\pm kr} \qquad k^2 = \frac{2\mu|E|}{\hbar^2}$$

solve.

For the region where $V(r)$ is small but non negligible, try.

$$U_E(r) = f(r)e^{\pm kr}$$

expect to be smooth, slowly varying; f'' negligible compared to f'

$$\frac{dU_E}{dr} = f'(r)e^{\pm kr} \pm kf(r)e^{\pm kr}$$

$$\frac{d^2U_E}{dr^2} = f''(r)e^{\pm kr} \pm 2kf'(r)e^{\pm kr} + k^2f(r)e^{\pm kr}$$

so, plugging into the Schrödinger equation

$$\cancel{f''(r)e^{\pm kr}} \pm 2kf'(r)e^{\pm kr} + \cancel{k^2f(r)e^{\pm kr}} = \cancel{k^2f(r)e^{\pm kr}} + \frac{2\mu}{\hbar^2}V(r)f(r)e^{\pm kr}$$

ignore "smooth"

so $$\frac{f'}{f} = \mp\frac{\mu}{k\hbar^2}V(r),$$

and $$f(r) = f(r_0)\, e^{\mp\frac{\mu}{k\hbar^2}\int_{r_0}^{r} V(r)dr}$$

$$\boxed{U_E(r) = \text{constant} \times e^{\mp\frac{\mu}{k\hbar^2}\int_{r_0}^{r} V(r)dr}\, e^{\pm kr} \qquad k = \frac{2\mu|E|}{\hbar^2}}$$

like pages 344–345 of text

Note for Coulomb, $V(r) = -\frac{Ze^2}{r}$, and

$$\int_{r_0}^{r} V(r)\,dr = -Ze^2(\ln r - \ln r_0)$$

and $U_E(r) = \text{constant} \times r^{\left(\frac{Z\mu e^2}{K\hbar^2}\right)} e^{\pm Kr}$

For potentials with $rV(r) \to 0$ as $r \to \infty$, the integral eventually becomes unimportant.

True bound states never involve the e^{+Kr} solution. Otherwise there would be an ∞ amount of wave function away from the origin... well this sort of is what happens.

The "discrete" energy levels that occur in bound states result from "matching" only the e^{-Kr} term in the tunneling region. Pictorially: