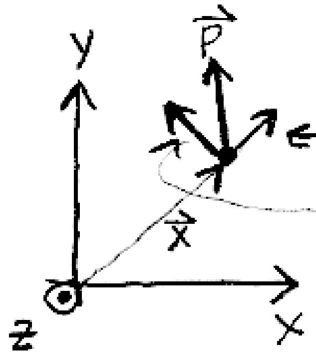
Angular Momentum in 3-d



$|\vec{x}|$ is ρ (in 2-d)

← "ρ" component of $\vec{p}$: $\vec{p}_\rho$

→ "ϕ" component of $\vec{p}$

<u>magnitude</u> is $\left|\left(\frac{\vec{x}}{\rho}\right) \times \vec{p}\right|$

reminder of 2-d

or $\left|\frac{L_z}{\rho}\right|$

note: $|\vec{p}|^2 = \underbrace{|\vec{p}_\rho|^2 + \frac{L_z^2}{\rho^2}}$ QM in coord rep.

$$-\hbar^2\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)$$

and $e^{\frac{-i\alpha L_z}{\hbar}} = \underset{\sim}{U}[R(\alpha\hat{k})]$

Quantum Unitary Operator that describes rotation <u>about</u> the $\hat{k}$ axis by an about ϕ (important to master this relationship).

note $\underset{\sim}{U}[R(\phi\hat{k})]|\psi\rangle \doteq e^{-\alpha\frac{\partial}{\partial\phi}}\,\psi(\rho,\phi)$

$$\doteq \psi(\rho,\phi) - \alpha\frac{\partial\psi}{\partial\phi} + \frac{1}{2}\alpha^2\frac{\partial^2\psi}{\partial\phi} - \frac{1}{3!}\alpha^3\frac{\partial^3\psi}{\partial\phi^3} + \cdots$$

$$\doteq \psi(\rho,\phi-\alpha)$$

→ could have picked any pair of axis in place of x, y

$x, y \rightarrow \hat{i} \times \hat{j} = \hat{k}$ so $\underset{\sim}{L}_z = \underset{\sim}{x} \underset{\sim}{p}_y - \underset{\sim}{y} \underset{\sim}{p}_x$ $\quad x \leftrightarrow 1$

$y, z \rightarrow \hat{j} \times \hat{k} = \hat{i}$ so $\underset{\sim}{L}_x = \underset{\sim}{y} \underset{\sim}{p}_z - \underset{\sim}{z} \underset{\sim}{p}_y$ $\quad y \leftrightarrow 2$

$z, x \rightarrow \hat{k} \times \hat{i} = \hat{j}$ so $\underset{\sim}{L}_y = \underset{\sim}{z} \underset{\sim}{p}_x - \underset{\sim}{x} \underset{\sim}{p}_z$ $\quad z \leftrightarrow 3$

as discussed earlier, $[\underset{\sim}{L}_i, \underset{\sim}{L}_j] = i\hbar \sum_k \varepsilon_{ijk} \underset{\sim}{L}_k$

$$\text{so, } [\underset{\sim}{L}_y, \underset{\sim}{L}_x] = [\underset{\sim}{L}_2, \underset{\sim}{L}_1] = i\hbar \underbrace{\varepsilon_{213}}_{-1} \underset{\sim}{L}_3$$

$$[\underset{\sim}{L}_y, \underset{\sim}{L}_x] = -i\hbar \underset{\sim}{L}_z$$

so imagine taking the cross product of the component unit vectors (($\hat{j}, \hat{i}$) above); take the cross product ($\hat{j} \times \hat{i} = -\hat{k}$) and the component on right hand side is the corresponding component ($-\underset{\sim}{L}_z$ or $-\underset{\sim}{L}_3$).

Operators that Describe Rotations

1) <u>about</u> z : $\underset{\sim}{U}[R(\alpha\hat{k})] = e^{-\frac{i}{\hbar}\alpha \underset{\sim}{L}_z}$

2) about x : $\underset{\sim}{U}[R(\alpha\hat{i})] = e^{-\frac{i}{\hbar}\alpha \underset{\sim}{L}_x}$

about y : $\underset{\sim}{U}[R(\alpha\hat{j})] = e^{-\frac{i}{\hbar}\alpha \underset{\sim}{L}_y}$

3) sequential rotations :

$$\underset{\sim}{U}[R(\alpha_x \hat{i})]\ \underset{\sim}{U}[R(\alpha_z \hat{k})] = \underbrace{e^{\frac{-i}{\hbar}\alpha_x \underset{\sim}{L}_x} e^{\frac{-i}{\hbar}\alpha_z \underset{\sim}{L}_z}}$$

first rotate about $\hat{k}$ by amount α_z

order of $\underset{\sim}{L}_x$ and $\underset{\sim}{L}_z$ matters since $[\underset{\sim}{L}_x, \underset{\sim}{L}_z] \neq 0$

$$\neq \underset{\sim}{U}[R(\alpha_z \hat{k})]\underset{\sim}{U}[R(\alpha_x \hat{i})] = e^{\frac{-i}{\hbar}\alpha_z \underset{\sim}{L}_z} e^{\frac{-i}{\hbar}\alpha_x \underset{\sim}{L}_x}$$

→ will do a demo with textbooks

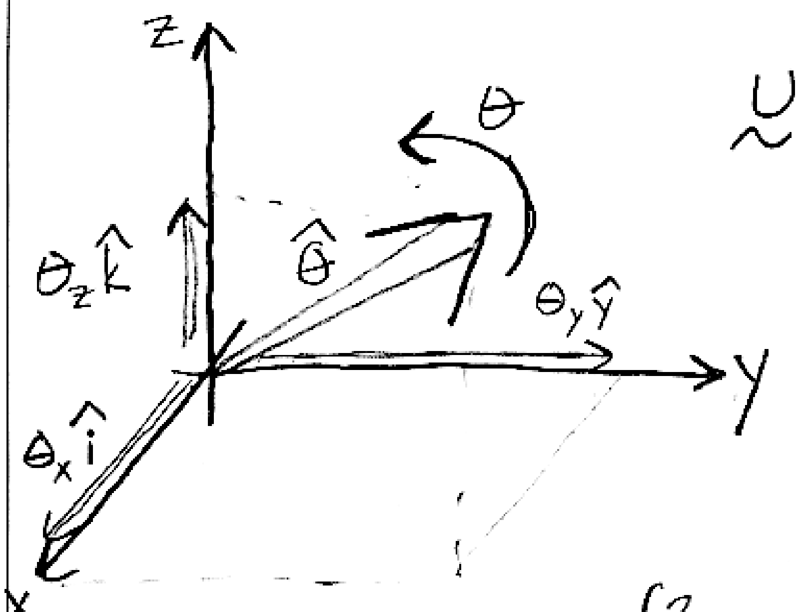
4) about an <u>arbitrary</u> direction $\hat{\theta}$ (nothing special about z, x, y)

$$\hat{\theta} = \theta_x \hat{i} + \theta_y \hat{j} + \theta_z \hat{k}$$

$$\hat{\theta}^2 = 1 = \theta_x^2 + \theta_y^2 + \theta_z^2$$

$$\hat{\theta}\cdot\vec{\underset{\sim}{L}} \equiv \theta_x \underset{\sim}{L}_x + \theta_y \underset{\sim}{L}_y + \theta_z \underset{\sim}{L}_z$$

by an amount θ $\left(\neq \sqrt{\theta_x^2+\theta_y^2+\theta_z^2}\right)$



$$\underset{\sim}{U}[R(\underbrace{\theta\hat{\theta}})] = e^{\frac{-i}{\hbar}\theta\hat{\theta}\cdot\vec{\underset{\sim}{L}}}$$

book calls θ

$$= e^{\frac{-i}{\hbar}\theta[\theta_x \underset{\sim}{L}_x + \theta_y \underset{\sim}{L}_y + \theta_z \underset{\sim}{L}_z]}$$

an arbitrary rotation can be described this way

(3 params: θ_x, θ_y, and θ)

Eigenvalue Problem... Spherical Symmetry

$$\underset{\sim}{H} = \frac{\vec{\underset{\sim}{P}}^2}{2\mu} + V(|\underset{\sim}{\vec{x}}|)$$

often; no dependendence on ϕ or θ

$\underset{\sim}{r} = |\underset{\sim}{\vec{x}}|$

m used for ϕ

$$= \frac{1}{2\mu}\left(\underset{\sim}{P_r}^2 + \frac{\underset{\sim}{L}^2}{\underset{\sim}{r}^2}\right) + V(r)$$

radial component of $\underset{\sim}{\vec{P}} = \dfrac{\underset{\sim}{x}\underset{\sim}{P_x} + \underset{\sim}{y}\underset{\sim}{P_y} + \underset{\sim}{z}\underset{\sim}{P_z}}{r}$

$$\underset{\sim}{L}^2 = \underset{\sim}{L_x}^2 + \underset{\sim}{L_y}^2 + \underset{\sim}{L_z}^2$$

Finding Eigenvalues of $\underset{\sim}{H}$ Boils Down to....

1) since $[\underset{\sim}{H}, \underset{\sim}{L_z}] = 0$ choose eigenkets of $\underset{\sim}{L_z}$

2) since $[\underset{\sim}{H}, \underset{\sim}{L}^2] = 0$ and $[\underset{\sim}{L}^2, \underset{\sim}{L_z}] = 0$ choose eigenkets of both $\underset{\sim}{L}^2$ and $\underset{\sim}{L_z}$

3) finally, will have to solve the "radial" equation $\left[\frac{1}{2\mu}\left(\underset{\sim}{P_r}^2 + \frac{\text{E.V. of } \underset{\sim}{L}^2}{\underset{\sim}{r}^2}\right) + V(r)\right]|\psi\rangle = E|\psi\rangle$

By representing into coordinate space, one finds:

$$\frac{1}{2\mu}\left(p_r^2 + \frac{1}{\underset{\sim}{r}^2}\underset{\sim}{L}^2 + V(\underset{\sim}{r})\right)|\psi\rangle = E|\psi\rangle$$

$$-\frac{\hbar^2}{2\mu}\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \underbrace{\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}}_{\text{already solved!}}\right) + V(r)$$

$$\frac{\partial^2}{\partial\phi^2}\Phi_m(\phi) = -m^2\Phi_m(\phi)$$

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}}e^{im\phi}$$

difficult to solve as a differential equation. Use raising & lowering technique

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$$\underset{\sim}{L}^2|\alpha\beta\rangle = \alpha|\alpha\beta\rangle \qquad (\alpha \text{ unknown})$$

$$\underset{\sim}{L}_z|\alpha\beta\rangle = \beta|\alpha\beta\rangle \qquad (\text{actually, } \beta \text{ is an integer})$$

note: $[\underset{\sim}{L}^2, \underset{\sim}{L}_j] = \sum_{i=1}^{3}[\underset{\sim}{L}_i^2, \underset{\sim}{L}_j]$

j is given

$$= \sum_{i=1}^{3}\left(\underset{\sim}{L}_i\underbrace{[\underset{\sim}{L}_i, \underset{\sim}{L}_j]}_{i\hbar\varepsilon_{ijk}\underset{\sim}{L}_k} + \underbrace{[\underset{\sim}{L}_i, \underset{\sim}{L}_j]}_{i\hbar\varepsilon_{ijk}\underset{\sim}{L}_k}\underset{\sim}{L}_i\right)$$

$$= i\hbar\sum_{i=1}^{3}\varepsilon_{ijk}\left(\underset{\sim}{L}_i\underset{\sim}{L}_k + \underset{\sim}{L}_k\underset{\sim}{L}_i\right)$$

k is a "tag along"
given $i \neq j$, k is fixed