## Translations in 2-dimensions (p. 306)

Because $[\underset{\sim}{P}_x, \underset{\sim}{P}_y] = 0$, if one wants to actively push a state forward by a vector displacement $\vec{a} = a_x \hat{x} + a_y \hat{y}$, one can do this sequentially:

$$\underset{\sim}{T}_x(a_x) \equiv e^{\frac{-ia_x}{\hbar} P_x} \qquad \underset{\sim}{T}_y(a_y) \equiv e^{\frac{-ia_y}{\hbar} P_y}$$

since $P_x P_y = P_y P_x$, $\underset{\sim}{T}_x(a_x)\, \underset{\sim}{T}_y(a_y) = \underset{\sim}{T}_y(a_y)\, \underset{\sim}{T}_x(a_x)$

just a power series in $P_x$

power series in $P_y$

to translate $|\psi\rangle$ actively by $a_x, a_y$:

$$= \underset{\sim}{T}_x(a_x)\underset{\sim}{T}_y(a_y)|\psi\rangle = \underset{\sim}{T}_y(a_y)\underset{\sim}{T}_x(a_x)|\psi\rangle$$

$$= e^{\frac{-ia_x}{\hbar}P_x} e^{\frac{-ia_y}{\hbar}P_y}|\psi\rangle = e^{\frac{-ia_y P_y}{\hbar}} e^{\frac{-ia_x P_x}{\hbar}}|\psi\rangle$$

note:

$$e^{\frac{-ia_x P_x}{\hbar}} e^{\frac{-ia_y P_y}{\hbar}} = \left(1 - \frac{ia_x P_x}{\hbar} + \frac{1}{2!}\frac{a_x^2 P_x^2}{\hbar^2} \ldots\right)\left(1 - \frac{ia_y P_y}{\hbar} + \frac{1}{2!}\frac{a_y^2 P_y^2}{\hbar^2} \ldots\right)$$

$$= 1 - \frac{ia_x P_x}{\hbar} - \frac{ia_y P_y}{\hbar} + \frac{1}{2!}\frac{1}{\hbar^2}\left(a_x^2 P_x^2 + 2a_x a_y P_x P_y + a_y^2 P_y^2\right) + \ldots$$

$$(a_x P_x + a_y P_y)^2 = a_x^2 P_x^2 + a_x a_y \underbrace{(P_x P_y + P_y P_x)}_{\text{since } [P_x, P_y] = 0} + a_y^2 P_y^2$$

this term $= 2P_x P_y = 2P_y P_x$

and so:

$$e^{-\frac{i a_x p_x}{\hbar}} e^{-\frac{i a_y p_y}{\hbar}} = 1 - \frac{i}{\hbar}(a_x p_x + a_y p_y) + \frac{1}{2!}\frac{i}{\hbar^2}(a_x p_x + a_y p_y)^2 + \cdots$$

$$= e^{-\frac{i}{\hbar}(a_x p_x + a_y p_y)}$$

] exponents add <u>only</u> when operators in exponents commute.

<u>Counterexample</u>:

$$e^{-ik\underset{\sim}{x}} e^{-\frac{ia}{\hbar}p} \rightarrow \underline{\text{cannot}} \text{ assume } = e^{-i(k\underset{\sim}{x} + \frac{a}{\hbar}p)}$$

this happens to be a special case that can be computed (see p. 478)

$$e^{-ik\underset{\sim}{x}} e^{-\frac{ia}{\hbar}p} = e^{\frac{ika}{2}} e^{-i(k\underset{\sim}{x} + \frac{a}{\hbar}p)}$$

But in <u>general</u>, for <u>arbitrary</u> $[\underset{\sim}{A}, \underset{\sim}{B}]$, there is no simple relationship between $e^{A}e^{B}$ and $e^{\underset{\sim}{A}+B}$.

<u>General Translation</u>

Active, described by $\vec{a}$, then

$$\vec{a}\cdot\vec{p} \equiv a_x p_x + a_y p_y + a_z p_z$$

$$\underset{\sim}{T}(\vec{a}) = e^{-\frac{i}{\hbar}\vec{a}\cdot\vec{p}}$$

<u>note</u>: <u>passive</u>: $\underset{\sim}{T}^{\dagger}(\vec{a})\,\underset{\sim}{x}\,\underset{\sim}{T}(\vec{a})$

$$= e^{\frac{i}{\hbar}(a_x p_x + a_y p_y + a_z p_z)}\,\underset{\sim}{x}\,e^{-\frac{i}{\hbar}(a_x p_x + a_y p_y + a_z p_z)}$$

since $p_y, p_z$ commute with $\underset{\sim}{x}$

$$\underset{\sim}{T}^{\dagger}(\vec{a})\,\underset{\sim}{x}\,\underset{\sim}{T}(\vec{a}) = e^{\frac{i}{\hbar}a_x p_x}\,\underset{\sim}{x}\,e^{\frac{-i}{\hbar}a_x p_x}$$

$$= e^{\frac{i}{\hbar}a_x p_x}\,\underset{\sim}{x}\left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-i}{\hbar}a_x p_x\right)^n\right]$$

want to work the $p_x^n$ by the $\underset{\sim}{x}$.

$$[\underset{\sim}{x}, p_x^n] = i\hbar n p^{n-1}$$ (derived long ago)

$$\underset{\sim}{x} p_x^n = p_x^n \underset{\sim}{x} + i\hbar n p^{n-1}$$

$$\underset{\sim}{T}^{\dagger}(\vec{a})\,\underset{\sim}{x}\,\underset{\sim}{T}(\vec{a}) = e^{\frac{i}{\hbar}a_x p_x}\left(\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-i}{\hbar}a_x p_x\right)^n\right)\underset{\sim}{x}$$

$$+\,e^{\frac{i}{\hbar}a_x p_x}\underbrace{\underbrace{\left(\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-i}{\hbar}a_x\right)i\hbar n\left(\frac{-i}{\hbar}a_x\right)^{n-1}\right)}_{a_x \sum_{n=1}^{\infty}\frac{1}{(n-1)!}\left(\frac{-i}{\hbar}a_x p_x\right)^{n-1}}}_{a_x e^{\frac{-i}{\hbar}a_x p_x}}$$

$$= e^{\frac{i}{\hbar}a_x p_x}e^{\frac{-i}{\hbar}a_x p_x}(\underset{\sim}{x} + a_x)$$

$$\left.\begin{aligned}
&\underset{\sim}{T}^{\dagger}(\vec{a})\,\underset{\sim}{x}\,\underset{\sim}{T}(\vec{a}) = \underset{\sim}{x} + a_x \\
&\text{similarly,} \\
&\underset{\sim}{T}^{\dagger}(\vec{a})\,\underset{\sim}{y}\,\underset{\sim}{T}(\vec{a}) = \underset{\sim}{y} + a_y \\
&\underset{\sim}{T}^{\dagger}(\vec{a})\,\underset{\sim}{z}\,\underset{\sim}{T}(\vec{a}) = \underset{\sim}{z} + a_z
\end{aligned}\right\}$$ consequence of commutation relations.

of course,

$$\underset{\sim}{T}(\vec{a})|\psi\rangle \doteq e^{\frac{-i}{\hbar}\left(a_x \frac{\partial}{\partial x} + a_y \frac{\partial}{\partial y} + a_z \frac{\partial}{\partial z}\right)} \psi(x,y,z)$$

$$\doteq \underbrace{\sum_{n l m} \left(\frac{-i}{\hbar}\right)^{n+l+m} \frac{a_x^{l} a_y^{m} a_z^{n}}{l!\,m!\,n!} \frac{\partial^{n+l+m}}{\partial x^n \partial y^l \partial z^m} \psi(x,y,z)}_{\text{power series, Taylor expansion about } \psi(x,y,z)}$$

$$\doteq \psi(x-a_x,\, y-a_y,\, z-a_z).$$

<u>Rotations</u>:

1) about z axis, Generator will be:

$$= \underset{\sim}{L}_z = \underset{\sim}{X} P_y - \underset{\sim}{Y} P_x$$

2) those about x axis, generator:

$$= \underset{\sim}{L}_x = \underset{\sim}{Y} P_z - \underset{\sim}{Z} P_y \quad \text{etcetera}$$

3) UNLIKE $[P_z, P_x] = 0$

$$[\underset{\sim}{L}_z, \underset{\sim}{L}_x] = [\underset{\sim}{X} P_y - \underset{\sim}{Y} P_x,\, \underset{\sim}{Y} P_z - \underset{\sim}{Z} P_y]$$

$$= [\underset{\sim}{X} P_y, \underset{\sim}{Y} P_z] + [\underset{\sim}{Y} P_x, \underset{\sim}{Z} P_y]$$

$$= \underset{\sim}{X}[P_y, \underset{\sim}{Y}] P_z + [\underset{\sim}{Y}, P_y] P_x \underset{\sim}{Z}$$

$$= -\underset{\sim}{X}(i\hbar) P_z + i\hbar P_x \underset{\sim}{Z}$$

$$[\underset{\sim}{L}_z, \underset{\sim}{L}_x] = i\hbar(\underset{\sim}{Z} P_x - \underset{\sim}{X} P_y) = i\hbar \underset{\sim}{L}_y$$

4) More generally $[\underset{\sim}{L}_i, \underset{\sim}{L}_j] = i\hbar \varepsilon_{ijk} \underset{\sim}{L}_k$

5) Consequence: eigenkets of $\underset{\sim}{L}_z$ not, in general, eigenkets of $\underset{\sim}{L}_y$ and/or $\underset{\sim}{L}_x$

6) When $\underset{\sim}{H}$ = invariant under all rotations it will mean: $0 = [\underset{\sim}{H}, \underset{\sim}{L}_z] = [\underset{\sim}{H}, \underset{\sim}{L}_y] = [\underset{\sim}{H}, \underset{\sim}{L}_x]$ and simultanews eigenkets of $\underset{\sim}{H}$ and one of $\underset{\sim}{L}_z$, $\underset{\sim}{L}_y$, or $\underset{\sim}{L}_x$ (usually $\underset{\sim}{L}_z$)

7) Once we get connection between rotations + $\underset{\sim}{L}_z$, we'll deduce eigenkets and eigenstates of $\underset{\sim}{L}_z$

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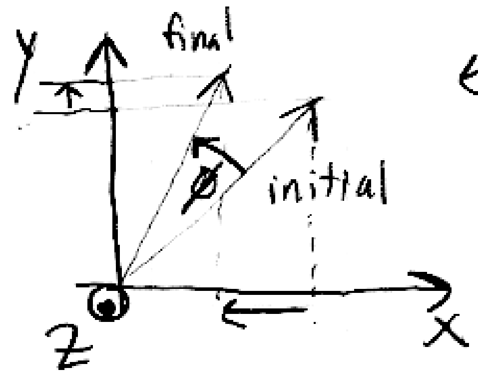
Rotations $\longleftrightarrow$ $\underset{\sim}{L}_z$

A) Active rotation by $\phi$ (counterclockwise) about z:




x component gets smaller
y component gets bigger.

$$\bar{x} = -y\sin\phi + x\cos\phi$$

$$\bar{y} = +x\sin\phi + y\cos\phi$$

change

important for $\phi = 0$

How we deduce the generator of rotations ?

want $\lim_{\phi \to 0} \underset{\sim}{U}(R(\phi \hat{k})) = \underset{\sim}{\mathbb{1}} - \frac{i\phi}{\hbar} \underset{\sim}{G}(\hat{k})$

since $\phi$ is small, call it $\lim_{\phi \to 0} \phi = \varepsilon_z$

Suppose: $\psi(x,y,z) = \langle x,y,z | \psi \rangle$

arbitrary state.

Want to analyze:

$$\underset{\sim}{U}(R(\varepsilon_z \hat{k}))|\psi\rangle = \iiint \underset{\sim}{U}(R(\varepsilon_z \hat{k}))|x,y,z\rangle\langle x,y,z|\psi\rangle dx dy dz$$

$$= \iiint |x - y\varepsilon_z, y + x\varepsilon_z, z\rangle \langle x,y,z|\psi\rangle dx dy dz$$

$$x' = x - y\varepsilon_z \quad y' = y + x\varepsilon_z \quad z' = z$$

$$dx dy dz = dx' dy' dz' \frac{\partial(x,y,z)}{\partial(x',y',z')}$$

$$\frac{\partial(x,y,z)}{\partial(x',y',z')} = \begin{vmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial y'} & \frac{\partial x}{\partial z'} \\ \frac{\partial y}{\partial x'} & \frac{\partial y}{\partial y'} & \frac{\partial y}{\partial z'} \\ \frac{\partial z}{\partial x'} & \frac{\partial z}{\partial y'} & \frac{\partial z}{\partial z'} \end{vmatrix} = \begin{vmatrix} 1 & \varepsilon_z & 0 \\ \varepsilon_z & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$x = x' + y'\varepsilon_z$$
$$y = y' - x'\varepsilon_z$$

$$= 1 - \varepsilon^2 \simeq 1$$

(still 1 to higher order in $\phi$)

or $\begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

similarly $\begin{pmatrix} \bar{P}_x \\ \bar{P}_y \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} P_x \\ P_y \end{pmatrix}$

B) Actively, then,

$$\underset{\sim}{U}(R(\phi\hat{k}))\,|x,y,z\rangle = |x\cos\phi - y\sin\phi,\ y\cos\phi + x\sin\phi,\ z\rangle$$

argument is <u>angle</u> times direction unit vector.. unit vector in z direction is $\hat{k}$

This is the analog of:

$$\underset{\sim}{T}(\vec{a})|x,y,z\rangle = |x+a_x, y+a_y, z+a_z\rangle$$

Which itself was the 3-d generalization of the 1-d designation.

$$\underset{\sim}{T}(a)\,|x\rangle = |x+a_x\rangle$$

C) Passively, requirement is:

$$\underset{\sim}{U}^{\dagger}(R(\phi\hat{k}))\,\underset{\sim}{x}\,\underset{\sim}{U}(R(\phi\hat{k})) = \underset{\sim}{x}\cos\phi - \underset{\sim}{y}\sin\phi$$

$$\underset{\sim}{U}^{\dagger}(R(\phi\hat{k}))\,\underset{\sim}{y}\,\underset{\sim}{U}(R(\phi\hat{k})) = \underset{\sim}{y}\cos\phi + \underset{\sim}{x}\sin\phi$$

$$\underset{\sim}{U}^{\dagger}(R(\phi\hat{k}))\,\underset{\sim}{z}\,\underset{\sim}{U}(R(\phi\hat{k})) = \underset{\sim}{z}$$

$$\underset{\sim}{U}(R(\varepsilon_z \hat{z}))|\psi\rangle = \iiint |x',y',z'\rangle\langle x'+\varepsilon_z y', y'-\varepsilon_z x', z'|\psi\rangle\, dx'dy'dz'$$

In other words,

$$\underset{\sim}{U}(R(\varepsilon_z \hat{k}))|\psi\rangle \doteq \psi(x+\varepsilon_z y, y-\varepsilon_z x, z)$$

$$\left(\underset{\sim}{\mathbb{1}} - \frac{i\varepsilon_z}{\hbar}\underset{\sim}{G}(\hat{k})\right)|\psi\rangle \doteq \psi(x,y,z) - \frac{i\varepsilon_z}{\hbar}\langle x,y,z|\underset{\sim}{G}(\hat{k})|\psi\rangle$$

so $\langle x,y,z|\underset{\sim}{G}(\hat{k})|\psi\rangle = i\hbar \lim\limits_{\varepsilon_z\to 0} \dfrac{\psi(x+\varepsilon_z y, y-\varepsilon_z x, z) - \psi(x,y,z)}{\varepsilon_z}$

$$= i\hbar\left( y\frac{\partial\psi}{\partial x} - x\frac{\partial\psi}{\partial y}\right)$$

$$\langle x,y,z|\underset{\sim}{G}(\hat{k})|\psi\rangle = x\frac{\hbar}{i}\frac{\partial\psi}{\partial y} - y\frac{\hbar}{i}\frac{\partial\psi}{\partial x}$$

so $\underset{\sim}{G}(\hat{k}) = \underset{\sim}{X}\,\underset{\sim}{P}_y - \underset{\sim}{Y}\,\underset{\sim}{P}_x = \underset{\sim}{L}_z$

## Change of Variables

$$\underset{\sim}{L}_z \doteq x\frac{\hbar}{i}\frac{\partial\psi}{\partial y} - y\frac{\hbar}{i}\frac{\partial\psi}{\partial x}$$

$x = \rho\cos\phi \qquad y = \rho\sin\phi$

$\dfrac{\partial x}{\partial\phi} = -\rho\sin\phi = -y \qquad \dfrac{\partial y}{\partial\phi} = \rho\cos\phi = x$

$$\frac{\partial}{\partial\phi} = \begin{pmatrix}\frac{\partial x}{\partial\phi} & \frac{\partial y}{\partial\phi}\end{pmatrix}\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{pmatrix} = \begin{pmatrix}-y & x\end{pmatrix}\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{pmatrix} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$