Translation invariance of $\underset{\sim}{H}$ ... (1-d)

means... $\langle\psi|\underset{\sim}{H}|\psi\rangle = \langle\psi_\varepsilon|\underset{\sim}{H}|\psi_\varepsilon\rangle = \langle\psi|\underset{\sim}{T}^\dagger(\varepsilon)\underset{\sim}{H}\underset{\sim}{T}(\varepsilon)|\psi\rangle$

or $\quad \underset{\sim}{H} = \underset{\sim}{T}^\dagger(\varepsilon)\underset{\sim}{H}\underset{\sim}{T}(\varepsilon)$

$$\underset{\sim}{T}(\varepsilon)\underset{\sim}{H} = \underbrace{\underset{\sim}{T}(\varepsilon)\underset{\sim}{T}^\dagger(\varepsilon)}_{\underset{\sim}{I}}\underset{\sim}{H}\underset{\sim}{T}(\varepsilon)$$

$$\underset{\sim}{T}(\varepsilon)\underset{\sim}{H} - \underset{\sim}{H}\underset{\sim}{T}(\varepsilon) = 0$$

$$[\underset{\sim}{T}(\varepsilon), \underset{\sim}{H}] = 0$$

expanding $\underset{\sim}{T}(\varepsilon) = \underset{\sim}{1} - \frac{i\varepsilon}{\hbar}\underset{\sim}{P} \Rightarrow [\underset{\sim}{1} - \frac{i\varepsilon}{\hbar}\underset{\sim}{P}, \underset{\sim}{H}] = 0$

or $\quad -\frac{i\varepsilon}{\hbar}[\underset{\sim}{P}, \underset{\sim}{H}] = 0$

$\langle\psi|\underset{\sim}{H}|\psi\rangle = \langle\psi_\varepsilon|\underset{\sim}{H}|\psi_\varepsilon\rangle \longrightarrow [\underset{\sim}{P}, \underset{\sim}{H}] = 0$

$\downarrow$ Ehrenfest's Theorem

$$\frac{d\langle \underset{\sim}{P}\rangle}{dt} = \frac{-i}{\hbar}\langle[\underset{\sim}{P}, \underset{\sim}{H}]\rangle = 0$$

$\langle \underset{\sim}{P}\rangle =$ constant, momentum conserved

__When__: $\underset{\sim}{H}$ independent of $\underset{\sim}{X}$

$[\underset{\sim}{P}, \underset{\sim}{H}] = 0$

Lagrangian Connection:

$$L = T - V = \frac{1}{2}m\dot{x}^2 - V(x)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \Rightarrow \frac{d}{dt}(\underbrace{m\dot{x}}_{p}) = -\frac{\partial V}{\partial x} = 0 \text{ when } V \text{ a constant}$$

$$\frac{dp}{dt} = 0$$

The Passive Viewpoint

$$\underset{\sim}{T}^{+}(\varepsilon)\,\underset{\sim}{X}\,\underset{\sim}{T}(\varepsilon) = \underset{\sim}{X} + \varepsilon\underset{\sim}{\mathbb{1}}$$

$$\underset{\sim}{T}^{+}(\varepsilon)\,P\,\underset{\sim}{T}(\varepsilon) = P$$

→ plug in the generator, $\underset{\sim}{T}(\varepsilon) = \underset{\sim}{\mathbb{1}} - \frac{i\varepsilon \underset{\sim}{G}}{\hbar}$

know $\underset{\sim}{T}^{+}(\varepsilon)\underset{\sim}{T}(\varepsilon) = \underset{\sim}{\mathbb{1}}$ means $\underset{\sim}{G} = \underset{\sim}{G}^{+}$

$$\left(\underset{\sim}{\mathbb{1}} - \frac{i\varepsilon}{\hbar}\underset{\sim}{G}\right)^{+}\underset{\sim}{X}\left(\underset{\sim}{\mathbb{1}} - \frac{i\varepsilon}{\hbar}\underset{\sim}{G}\right) = \underset{\sim}{X} + \varepsilon\underset{\sim}{\mathbb{1}}$$

$$\left(\underset{\sim}{\mathbb{1}} + \frac{i\varepsilon}{\hbar}\underset{\sim}{G}\right)\underset{\sim}{X}\left(\underset{\sim}{\mathbb{1}} - \frac{i\varepsilon}{\hbar}\underset{\sim}{G}\right) = \text{"}$$

$$\cancel{\underset{\sim}{X}} + \frac{i\varepsilon}{\hbar}\underset{\sim}{G}\underset{\sim}{X} - \frac{i\varepsilon}{\hbar}\underset{\sim}{X}\underset{\sim}{G} + \underbrace{\cancel{\frac{\varepsilon^2}{\hbar^2}}\underset{\sim}{G}\underset{\sim}{X}\underset{\sim}{G}}_{\text{neglect}} = \cancel{\underset{\sim}{X}} + \varepsilon\underset{\sim}{\mathbb{1}}$$

$$-\frac{i\varepsilon}{\hbar}[\underset{\sim}{X}, \underset{\sim}{G}] = \varepsilon\underset{\sim}{\mathbb{1}}$$

$$[\underset{\sim}{X}, \underset{\sim}{G}] = i\hbar\underset{\sim}{\mathbb{1}} = i\hbar$$

$$\underset{\sim}{G} = P + f(\underset{\sim}{X}) \quad \text{since } [\underset{\sim}{X}, f(\underset{\sim}{X})] = 0$$

Constrain $f(\underset{\sim}{X})$ with the P equation:

$$\left(\underset{\sim}{\mathbb{1}} + \frac{i\varepsilon}{\hbar}P + \frac{i\varepsilon}{\hbar}\underbrace{f^{*}(\underset{\sim}{X})}_{=f(\underset{\sim}{X}),\ \text{to keep } \underset{\sim}{G} \text{ Hermitian}}\right)P\left(\underset{\sim}{\mathbb{1}} - \frac{i\varepsilon}{\hbar}P - \frac{i\varepsilon}{\hbar}f(\underset{\sim}{X})\right) = P$$

ignoring $O(\varepsilon^2)$,

$$\cancel{P} + \cancel{\frac{i\varepsilon}{\hbar}P^2} - \cancel{\frac{i\varepsilon}{\hbar}P^2} + \frac{i\varepsilon}{\hbar}f(\underset{\sim}{X})P - \frac{i\varepsilon}{\hbar}P f(\underset{\sim}{X}) = \cancel{P}$$

$$[f(\underset{\sim}{X}), P] = 0 = i\hbar\frac{df}{d\underset{\sim}{X}} = 0$$

$f(\underset{\sim}{X}) = \text{constant}$

Finite Translations

Change notation: $\varepsilon \to a$ $\quad$ a never $\to 0$

View finite translations as a product of infinitesimals...

$$\underset{\sim}{T}(a) = \underset{\sim}{T}\left(\tfrac{a}{2}\right)\underset{\sim}{T}\left(\tfrac{a}{2}\right) = \underset{\sim}{T}\left(\tfrac{a}{3}\right)\underset{\sim}{T}\left(\tfrac{a}{3}\right)\underset{\sim}{T}\left(\tfrac{a}{3}\right)$$

$$= \underset{\sim}{T}\left(\tfrac{a}{4}\right)\underset{\sim}{T}\left(\tfrac{a}{4}\right)\underset{\sim}{T}\left(\tfrac{a}{4}\right)\underset{\sim}{T}\left(\tfrac{a}{4}\right)$$

$$\vdots$$

$$= \lim_{N\to\infty}\left[\underset{\sim}{T}\left(\tfrac{a}{N}\right)\right]^N$$

now this is infinitesimal

$$\underset{\sim}{T}\left(\tfrac{a}{N}\right) = \left(\mathbb{1} - \frac{i}{\hbar}\frac{a}{N}p\right)$$

so
$$\underset{\sim}{T}(a) = \lim_{N\to\infty}\left(\mathbb{1} - \underbrace{\frac{i}{\hbar}\frac{a}{N}}_{\substack{\text{commutes with } p \\ \text{aka independent of } \underset{\sim}{x}}} p\right)^N = e^{-\frac{i}{\hbar}ap}$$

$$\langle x|\underset{\sim}{T}(a)|\psi\rangle = e^{-\frac{i}{\hbar}a\frac{\hbar}{i}\frac{d}{dx}}\psi(x)$$

$$= e^{-a\frac{d}{dx}}\psi(x)$$

$$= \underbrace{\left(1 - a\frac{d}{dx} + \frac{1}{2!}a^2\frac{d^2}{dx^2} - \frac{1}{3!}a^3\frac{d^3}{dx^3} + \dots\right)\psi(x)}_{\text{Taylor expansion}}$$

$$= \psi(x-a)$$ (must overcome a).

$$\underset{\sim}{T}(a)\underset{\sim}{T}(b) = \underset{\uparrow}{e^{-\frac{i}{\hbar}ap}}\,\underset{\uparrow}{e^{-\frac{i}{\hbar}bp}} = e^{-\frac{i}{\hbar}(a+b)p} = \underset{\sim}{T}(a+b)$$

these commute

Physical Meaning of Translation Invariance

more or less.... experiments done:

- → My Lab
- → Lab at a competitor's place
- → Lab on moon
- → Lab in a distant galaxy

should all give the same results, like, the ionization energy of H is 13.6 eV.

Careful: translate only H atoms (system) not apparatus ("universe") won't get any results.

Reality: set up experiments so lots of things (elevators, smokers, janitors) don't influence.

Assumption of Translation Invariance IS THE BASIS OF ASTROPHYSICS...

otherwise we couldn't assume much about physics occurring in other galaxies..

Time Translation Invariance

idea:

$$|\psi_0\rangle = |\psi(t_1)\rangle \rightarrow \underset{t_1 \rightarrow t_1+\varepsilon}{\text{increment time by } \varepsilon} \rightarrow |\psi(t_1+\varepsilon)\rangle$$

$t_1 \neq t_2$

when will these be equal?

$$|\psi_0\rangle = |\psi(t_2)\rangle \rightarrow t_2 \rightarrow t_2+\varepsilon \rightarrow |\psi(t_2+\varepsilon)\rangle$$

$i\hbar \frac{d}{dt}|\psi(t_1)\rangle = \underset{\sim}{H}(t_1)|\psi(t_1)\rangle \qquad i\hbar\frac{d}{dt}|\psi(t_2)\rangle = \underset{\sim}{H}(t_2)|\psi(t_2)\rangle$

$|\psi(t_1+\varepsilon)\rangle \simeq |\psi(t_1)\rangle + \varepsilon\frac{d}{dt}|\psi(t_1)\rangle \qquad |\psi(t_2+\varepsilon)\rangle \simeq |\psi(t_2)\rangle + \varepsilon\frac{d}{dt}|\psi(t_2)\rangle$

$\simeq \underline{|\psi_0\rangle + \frac{\varepsilon}{i\hbar}\underset{\sim}{H}(t_1)|\psi_0\rangle} \qquad \simeq \underline{|\psi_0\rangle + \frac{\varepsilon}{i\hbar}\underset{\sim}{H}(t_2)|\psi_0\rangle}$

Equal when $\underset{\sim}{H}(t_1) = \underset{\sim}{H}(t_2)$

true when $\frac{dH}{dt} = 0$

in which case: $\left\langle \frac{d\underset{\sim}{H}}{dt}\right\rangle = \langle[\underset{\sim}{H},\underset{\sim}{H}]\rangle$ (Ehrenfest)

$\langle \underset{\sim}{H}\rangle$ = constant, Energy conserved

(Time Invariance of $\underset{\sim}{H}$) $\Leftrightarrow$ (Energy conserved)

Parity

$\left.\begin{array}{l} x \longrightarrow -x \\ p \longrightarrow -p \end{array}\right\}$ classically, using Hamiltonian

Quantum Mechanically: Parity operator $\underset{\sim}{\Pi}$

$\underset{\sim}{\Pi}|x\rangle = |-x\rangle$

eigenket of position, eigenvalue x

$\underset{\sim}{X}|x\rangle = x|x\rangle$

$\underset{\sim}{X}|-x\rangle = -x|-x\rangle$

$\underset{\sim}{\Pi}|p\rangle = |-p\rangle$

What about action on an arbitrary ket?

$$\underset{\sim}{\Pi}|\psi\rangle = \int_{-\infty}^{\infty} dx' \, \underset{\sim}{\Pi}|x'\rangle\langle x'|\psi\rangle = \int_{-\infty}^{\infty} dx' |-x'\rangle\langle x'|\psi\rangle$$

$$\underset{\sim}{\mathbb{1}} = \int dx' |x'\rangle\langle x'|$$

$x'' = -x'$
$dx'' = -dx'$

$$= -\int_{\infty}^{-\infty} dx'' |x''\rangle\langle -x''|\psi\rangle$$

$$\underset{\sim}{\Pi}|\psi\rangle = +\int_{-\infty}^{\infty} dx'' |x''\rangle\langle -x''|\psi\rangle$$

$$\langle x|\underset{\sim}{\Pi}|\psi\rangle = \int_{-\infty}^{\infty} dx'' \underbrace{\langle x|x''\rangle}_{\delta(x-x'')}\langle -x''|\psi\rangle$$

$$= \langle -x|\psi\rangle = \psi(-x)$$

in other words, when $\langle x|\psi\rangle = \psi(x)$

$$\langle x|\underset{\sim}{\Pi}|\psi\rangle = \psi(-x)$$

Sometimes say: "$\underset{\sim}{\Pi}\psi(x) = \psi(-x)$"

also, $\underset{\sim}{\Pi}\psi(p) = \psi(-p)$

$$\underset{\sim}{\Pi}^2|x\rangle = \underset{\sim}{\Pi}|-x\rangle = |x\rangle$$

$\underset{\sim}{\Pi}^2 = \underset{\sim}{\mathbb{1}}$ so $\boxed{\underset{\sim}{\Pi}^{-1} = \underset{\sim}{\Pi}}$

eigenvalues: $\underset{\sim}{\Pi}|\pi\rangle = \pi|\pi\rangle$

$$\underset{\sim}{\Pi}^2|\pi\rangle = \pi^2|\pi\rangle = \underset{\sim}{\mathbb{1}}\cdot|\pi\rangle$$

$\pi^2 = 1$, $\boxed{\pi = \pm 1}$

What bra corresponds to $\underset{\sim}{\Pi}|x\rangle$?

$|-x\rangle$

it's $\langle -x| \equiv \langle x|\underset{\sim}{\Pi}$

$$\boxed{\text{By definition, } \underset{\sim}{\Pi} = \underset{\sim}{\Pi}^{+} = \underset{\sim}{\Pi}^{-1} \text{ (Hermitian \underline{and} Unitary). eigenvalues are } \pm 1}$$

$$\underset{\sim}{\Pi}^{+}\underset{\sim}{X}\underset{\sim}{\Pi} \rightarrow \langle x'|\underset{\sim}{\Pi}^{+}\underset{\sim}{X}\underset{\sim}{\Pi}|x\rangle$$

$$= \langle -x'|\underset{\sim}{X}|-x\rangle = -x\langle -x'|-x\rangle$$

$$= -x\,\delta(-x'-(-x))$$

$$= -x\,\delta(x-x') = -x\,\delta(x'-x)$$

$$= -\langle x'|\underset{\sim}{X}|x\rangle = \langle x'|(-\underset{\sim}{X})|x\rangle$$

so $\underset{\sim}{\Pi}^{+}\underset{\sim}{X}\underset{\sim}{\Pi} = -\underset{\sim}{X}$ } <u>note</u> $\underset{\sim}{\Pi}^{+}\underset{\sim}{X}\underset{\sim}{\Pi} = -\underset{\sim}{X}$

$$\underset{\sim}{\Pi}\underset{\sim}{X}\underset{\sim}{\Pi} = -\underset{\sim}{X}$$

$$\cancel{\underset{\sim}{\Pi}\underset{\sim}{\Pi}}\underset{\sim}{X}\underset{\sim}{\Pi} = -\underset{\sim}{\Pi}\underset{\sim}{X}$$

$$\underset{\sim}{X}\underset{\sim}{\Pi} + \underset{\sim}{\Pi}\underset{\sim}{X} = 0$$

1) $\underset{\sim}{\Pi}$ does <u>not</u> commute with $\underset{\sim}{X}$ (or $\underset{\sim}{P}$)

2) $[\underset{\sim}{X}, \underset{\sim}{\Pi}]_{+} = 0$

<u>Hamiltonian</u>

Parity "invariant" if $\underset{\sim}{\Pi}^{+}\underset{\sim}{H}(\underset{\sim}{X}, \underset{\sim}{P})\underset{\sim}{\Pi} = \underset{\sim}{H}(\underset{\sim}{X}, \underset{\sim}{P})$

or $\quad \underset{\sim}{H}(-\underset{\sim}{x},-p) = \underset{\sim}{H}(\underset{\sim}{x},p)$

<u>Examples</u>

$$\underset{\sim}{H}(\underset{\sim}{x},p) = \frac{p^2}{2m} \Rightarrow \underset{\sim}{H}(-\underset{\sim}{x},-p) = \frac{(-p)^2}{2m}$$

$$= \frac{p^2}{2m} = \underset{\sim}{H}(\underset{\sim}{x},p)$$

$\rightarrow$ parity invariant

$$\underset{\sim}{H}(\underset{\sim}{x},p) = \frac{p^2}{2m} + V(\underset{\sim}{x})$$

$$\underset{\sim}{H}(-\underset{\sim}{x},-p) = \frac{(-p)^2}{2m} + V(-\underset{\sim}{x}) = \frac{p^2}{2m} + V(-\underset{\sim}{x})$$

$\Rightarrow$ will be parity invariant when $V(-\underset{\sim}{x}) = V(\underset{\sim}{x})$

[not $V(-\underset{\sim}{x}) = -V(-\underset{\sim}{x})$]

$\Rightarrow$ examples:

1-d $\quad V(\underset{\sim}{x}) = \frac{1}{2}k\underset{\sim}{x}^2$

$$V(\underset{\sim}{x}) = \begin{cases} 0 & |x| < L/2 \\ \infty & |x| > L/2 \end{cases}$$

3-d $\quad V(\vec{\underset{\sim}{x}}) = \frac{1}{|\vec{x}|}$

<u>When</u>: $\quad \underset{\sim}{\Pi}^{+}\underset{\sim}{H}(\underset{\sim}{x},p)\underset{\sim}{\Pi} = \underset{\sim}{H}(\underset{\sim}{x},p)$

$$\underset{\sim}{H}\underset{\sim}{\Pi} = \underset{\sim}{\Pi}\underset{\sim}{H} \Rightarrow [\underset{\sim}{\Pi},\underset{\sim}{H}] = 0$$