Consider: $|34\rangle \rightarrow |\langle x_1 x_2|34\rangle|^2 = \psi_3^2(x_1)\psi_4^2(x_2)$

one particle $n=3$ another $n=4$

BOSONS: $|34,S\rangle \propto |34\rangle + |43\rangle$

normalize: $\{\langle 34| + \langle 43|\}\{|34\rangle + |43\rangle\}$

$= \langle 34|34\rangle + \langle 43|43\rangle + \langle 34|43\rangle + \langle 43|34\rangle$

$= \langle 3|3\rangle\langle 4|4\rangle + \langle 4|4\rangle\langle 3|3\rangle + \langle 3|4\rangle\langle 4|3\rangle + \langle 4|3\rangle\langle 3|4\rangle$

$= 1\cdot 1 + 1\cdot 1 + 0\cdot 0 + 0\cdot 0$

$= 2$

and so: $|34,S\rangle = \frac{1}{\sqrt{2}}\left[|34\rangle + |43\rangle\right]$

FERMIONS: $|34,A\rangle = \frac{1}{\sqrt{2}}\left[|34\rangle - |43\rangle\right]$

How could you tell which?

$\Rightarrow$ project on to position basis

Very confusing factors of 2....

$|x_1 x_2\rangle \rightarrow$ product state, not possible for identical particles.

on to:

$$|x_1, x_2, S\rangle = \frac{1}{\sqrt{2}}\left[|x_1, x_2\rangle + |x_2 x_1\rangle\right] \quad \left(\begin{matrix}\text{Symmetric}\\ \text{case}\end{matrix}\right)$$

(state where $x_1 = x_2$ infinitesimal)

Imagine:

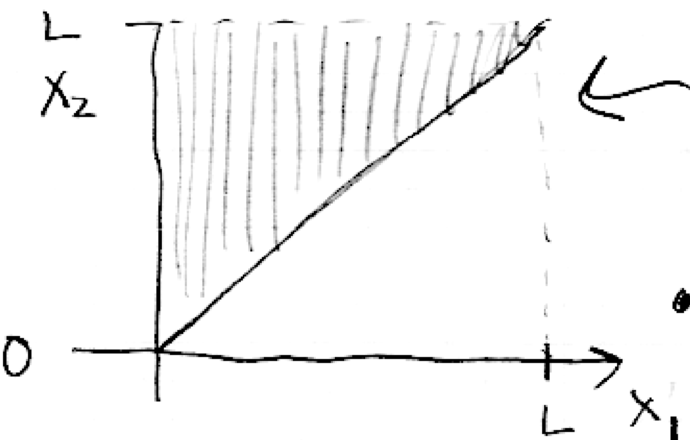
$$P_s(x_1, x_2) = |\langle x_1 x_2, S|34, S\rangle|^2 = P_s(x_2, x_1)$$

since $\langle x_1 x_2, S| = \langle x_2 x_1, S|$

Normalization Issue:

$$\int_{\substack{x_1 \text{ distinct} \\ \text{from } x_2}} dx_1\, dx_2\, P_s(x_1, x_2) = 1$$

(x_1 distinct from x_2) → one way: $\int_0^L dx_1 \int_{x_1}^L dx_2$



- area of integration is $\frac{1}{2}$ of total
- since integrand $P(x_1, x_2) = P(x_2, x_1)$

so

$$\int_{\substack{x_1 \text{ distinct} \\ \text{from } x_2}} dx_1\, dx_2\, P_s(x_1, x_2) = \frac{1}{2}\int_0^L \int_0^L dx_1\, dx_2\, P_s(x_1, x_2)$$

$$= 1$$

Convenient then to define:

$$\psi_s(x_1, x_2) = \frac{1}{\sqrt{2}} \langle x_1 x_2, S | 34, s \rangle$$

so $\displaystyle\int_0^L\int_0^L dx_1\, dx_2\, |\psi_s(x_1, x_2)|^2 = 1$

<u>but</u> <u>note</u> $P(x_1, x_2) = 2|\psi_s(x_1, x_2)|^2$

(really, issue is integration region)

$$\psi_s(x_1, x_2) = \frac{1}{\sqrt{2}} \langle x_1 x_2, S | 34, s \rangle$$

$$= \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\left(\langle x_1 x_2| + \langle x_2 x_1|\right)|34, s\rangle\right]$$

$$= \frac{1}{2}\left[\langle x_1 x_2 | 34, s\rangle + \langle x_2 x_1 | 34, s\rangle\right]$$

but $\langle x_1 x_2 | 34, s\rangle = \langle x_2 x_1 | 34, s\rangle$ → because of symmetrization of $|34, s\rangle$

so $\psi_s(x_1, x_2) = \langle x_1 x_2 | 34, s\rangle$

$$= \langle x_1 x_2 | \left\{\frac{1}{\sqrt{2}}\left(|34\rangle + |43\rangle\right)\right\}$$

$$= \frac{1}{\sqrt{2}}\left\{\langle x_1 | 3\rangle\langle x_2 | 4\rangle + \langle x_1 | 4\rangle\langle x_2 | 3\rangle\right\}$$

$$\psi_s(x_1, x_2) = \frac{1}{\sqrt{2}}\left(\psi_3(x_1)\psi_4(x_2) + \psi_3(x_2)\psi_4(x_1)\right)$$

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$ (Real Valued)

Similarly:

$$\psi_A(x_1, x_2) = \frac{1}{\sqrt{2}}\left(\psi_3(x_1)\psi_4(x_2) - \psi_3(x_2)\psi_4(x_1)\right)$$

How to Tell:

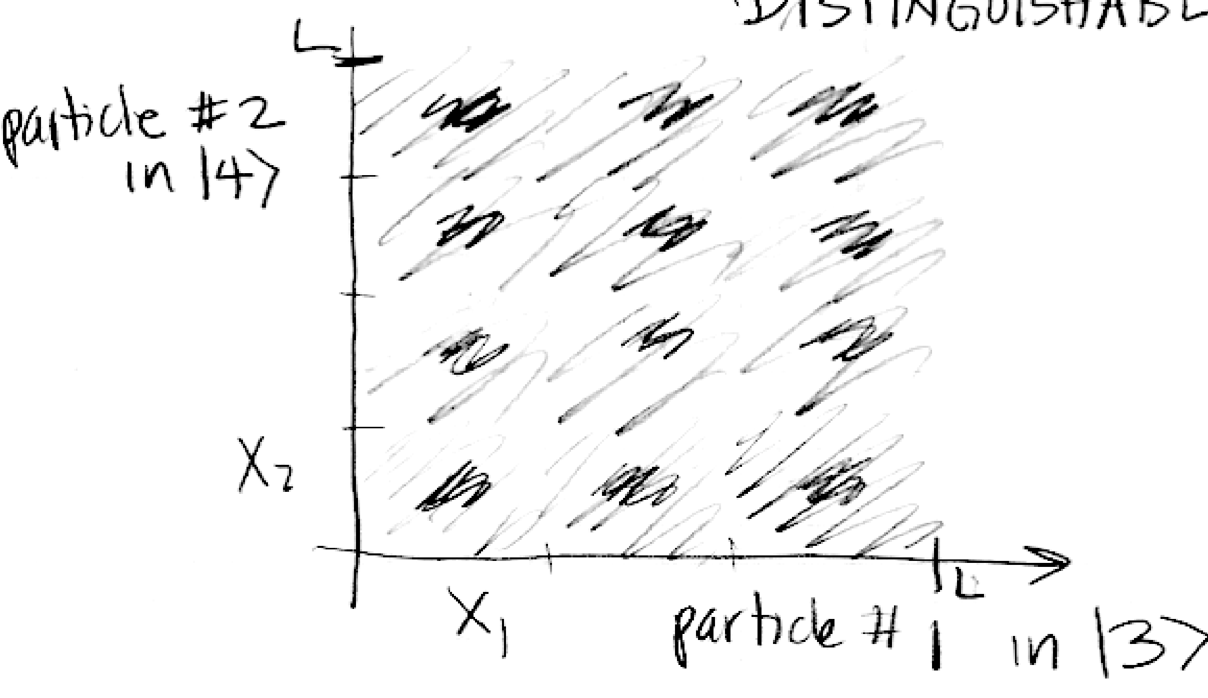
$$|\psi_{S/A}(x_1, x_2)|^2 = \frac{1}{2}\left[\psi_3^2(x_1)\psi_4^2(x_2) + \psi_3^2(x_2)\psi_4^2(x_2)\right.$$
$$\left.\pm 2\,\mathrm{Re}\left[\psi_3(x_1)\psi_3(x_2)\psi_4(x_1)\psi_4(x_2)\right]\right]$$

These are not the same! Most obviously at $x_1 = x_2$

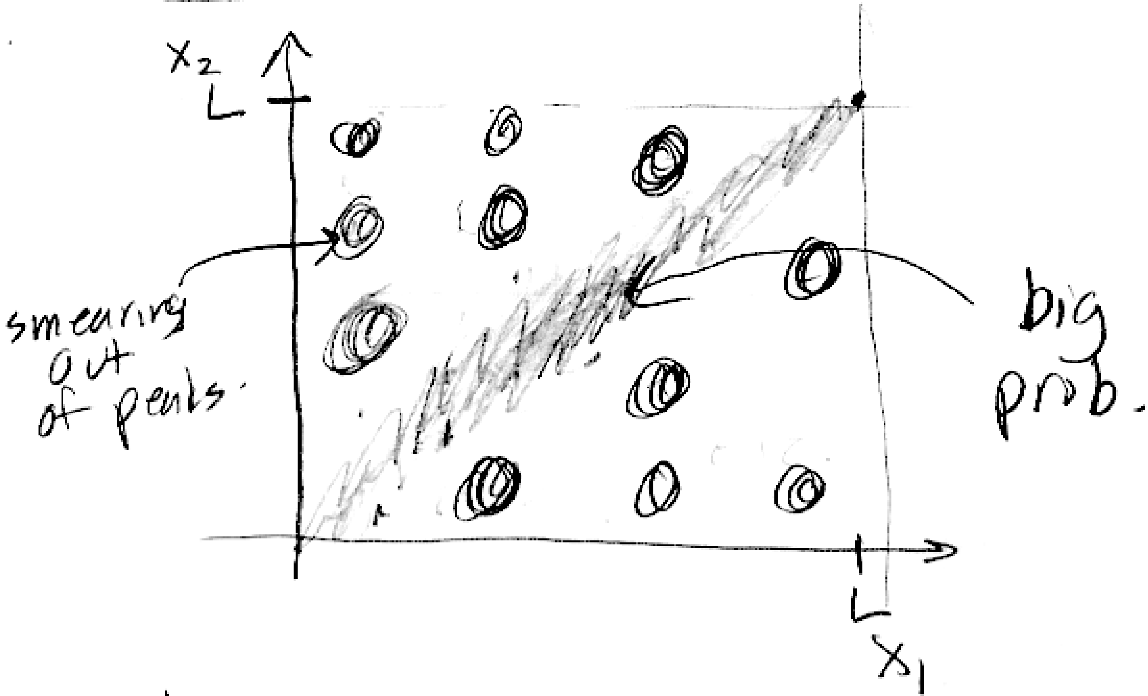
$$|\psi_S(x_1, x_1)|^2 = 2\psi_3^2(x_1)\psi_4^2(x_1)$$

$$|\psi_A(x_1, x_2)|^2 = 0$$

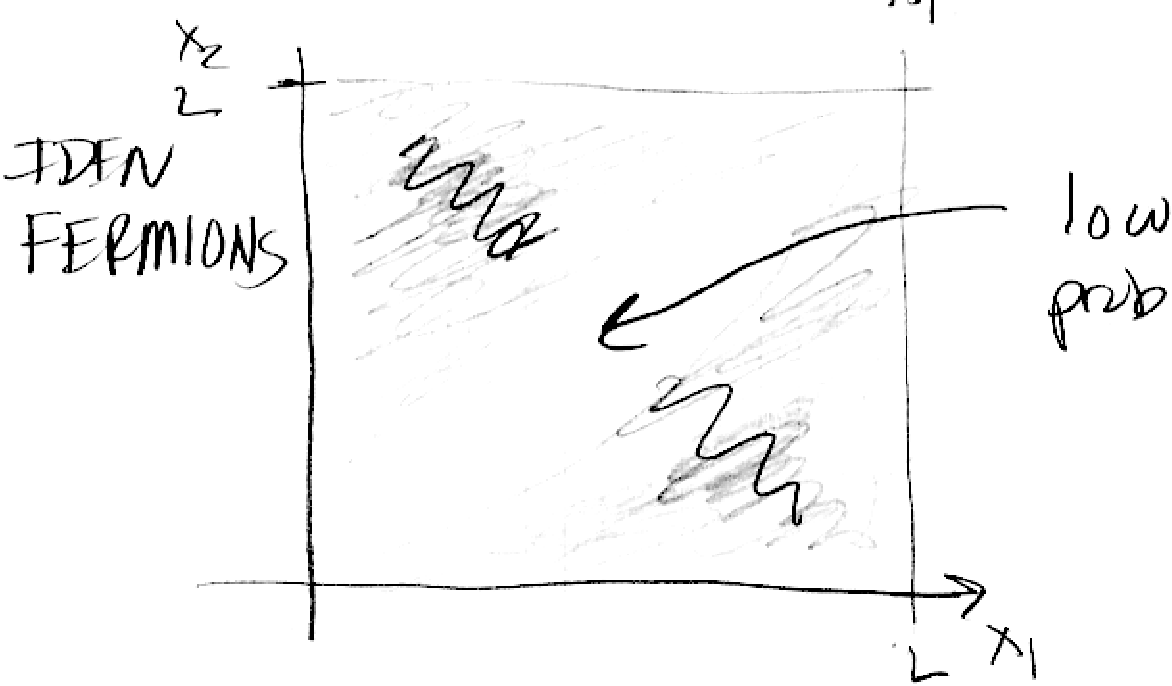
DISTINGUISHABLE



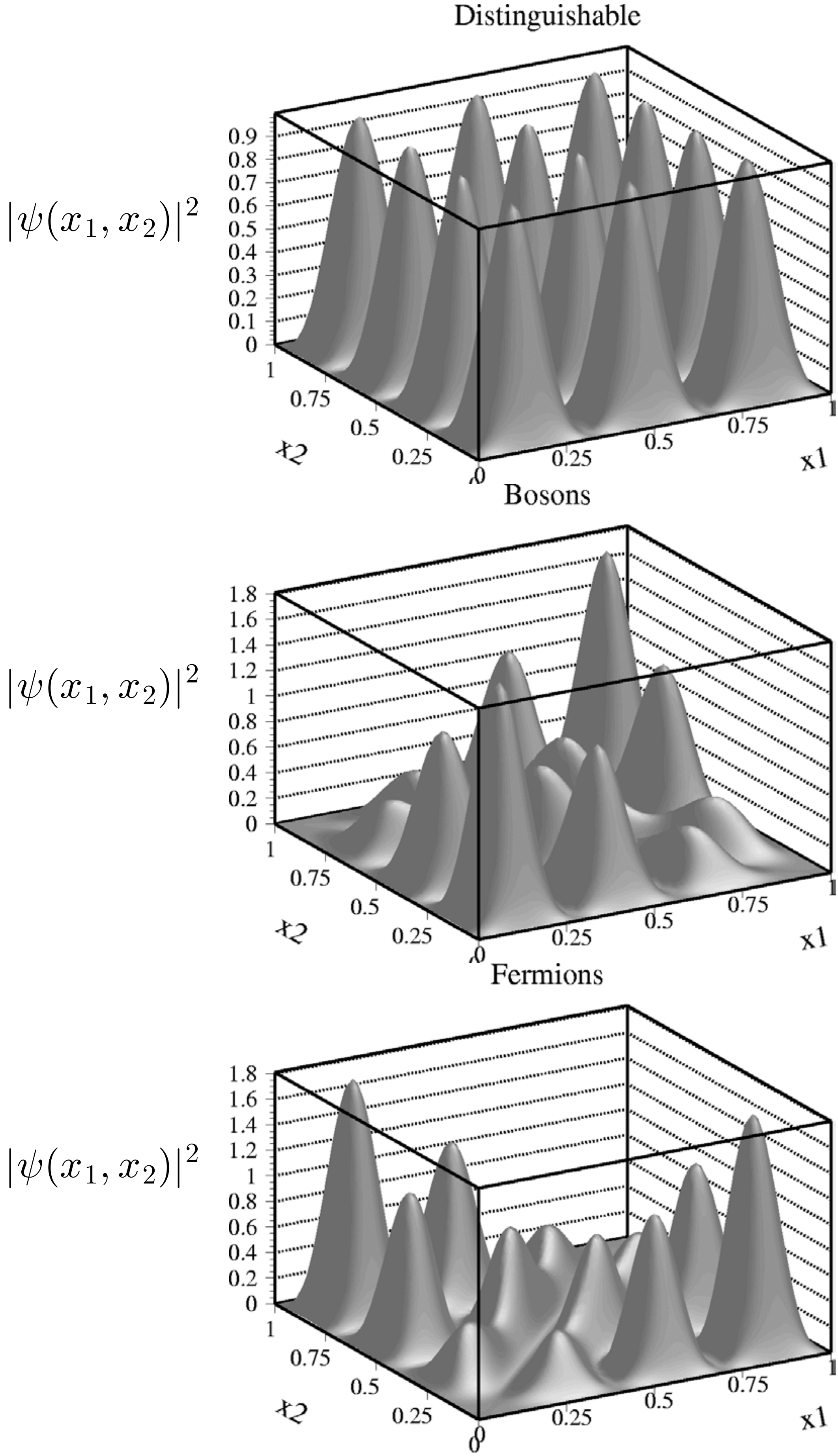
IDENT. **BOSONS**: INDISTINGUISHABLE



IDEN
FERMIONS



Dissatisfaction with my poor drawings – made better ones.


Distinguishable
$|\psi(x_1, x_2)|^2$
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1
0.75
0.5
0.25
x2
0
0.25
0.5
0.75
1
x1
Bosons
$|\psi(x_1, x_2)|^2$
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
1
0.75
0.5
0.25
x2
0
0.25
0.5
0.75
1
x1
Fermions
$|\psi(x_1, x_2)|^2$
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
1
0.75
0.5
0.25
x2
0
0.25
0.5
0.75
1
x1

Fermion Wavefunction:

"Slater Determinant"

→ states

$$\frac{1}{\sqrt{2}} \begin{vmatrix} \psi_3(x_1) & \psi_4(x_1) \\ \psi_3(x_2) & \psi_4(x_2) \end{vmatrix} = \psi_A(x_1, x_2)$$

↓ coordinates

3 Particles: enumerate permutations.

$$\Big[|n_1 n_2 n_3\rangle \pm |n_1 n_3 n_2\rangle + |n_2 n_3 n_1\rangle \pm |n_2 n_1 n_3\rangle$$

S: +
A: −

$$+ |n_3 n_1 n_2\rangle \pm |n_3 n_2 n_1\rangle \Big] \times \frac{1}{\sqrt{3 \cdot 2 \cdot 1}}$$

($3 \cdot 2 \cdot 1 = 3!$ permutations when $n_1 \neq n_2 \neq n_3$, $n_1 \neq n_3$

only correct for $n_1 \neq n_2 \neq n_3$ $n_1 \neq n_3$

a.k.a (fermions)

$$\frac{1}{\sqrt{3!}} \begin{vmatrix} \psi_{n_1}(x_1) & \psi_{n_2}(x_1) & \psi_{n_3}(x_1) \\ \psi_{n_1}(x_2) & \psi_{n_2}(x_2) & \psi_{n_3}(x_2) \\ \psi_{n_1}(x_3) & \psi_{n_2}(x_3) & \psi_{n_3}(x_3) \end{vmatrix}$$

4 particles: $+$ = symmetric $-$ = antisymmetric

$$\Big[|n_1 n_2 n_3 n_4\rangle \pm |n_1 n_2 n_4 n_3\rangle + |n_1 n_3 n_4 n_2\rangle \pm |n_2 n_3 n_4 n_1\rangle$$

12 = 4!

$$+ |n_2 n_3 n_1 n_4\rangle \pm |n_2 n_4 n_1 n_3\rangle + |n_3 n_4 n_1 n_2\rangle \pm |n_3 n_4 n_2 n_1\rangle$$

$$+ |n_3 n_1 n_2 n_4\rangle \pm |n_4 n_1 n_2 n_3\rangle + |n_4 n_1 n_3 n_2\rangle \pm |n_4 n_2 n_3 n_1\rangle \Big]$$

$$\times \frac{1}{\sqrt{4!}}$$

again, all n's must be different.

When 3 or more particles are involved, the dimensionality of the product subspace synthesized out of N single-particle states can be far greater than the dimensionality of the space allowed for identical particles. For example, when all N single particle states are distinct (like square well with $n_1 \neq n_2$, $n_1 \neq n_3$, $n_1 \neq n_4$, etc.) then the product subspace will have dimensionality

N!: $| \; \; \; \; \; \; \; \ldots \; \rangle$

N choices, N-1 choices, N-2, ..., 1 choice.

But: Identical Bosons: acceptable state must be symmetric with respect to EXCHANGE of all quantum #s belonging to every pair of identical particles

(refer to 4 particle)

only 1 ≠ N!

Identical Fermions : acceptable state must be antisymmetric with respect to EXCHANGE of all quantum #s belonging to every pair of identical particles   only 1 of N!

In 1-d, like square well, no other quantum #'s other than n.

In 3-d, at least one other: spin.

Pauli Principle :

2 electrons cannot occupy same "orbital" state (described by n, l, m) and same spin state.

2 electrons can share (n, l, m) as long as spin different, in which case the spin state must be

$$\propto \; |+-\rangle - |-+\rangle$$

↑ + not allowed!