Uncertainty + Bound States p.341

The uncertainty principle (or, that particles are waves) keeps the world from collapsing.

$$\underset{\sim}{H} = \underbrace{\frac{p_x^2 + p_y^2 + p_z^2}{2m}}_{\substack{\text{kinetic energy} \\ \text{of } m = \text{electron mass} \\ \text{assuming proton } \infty \text{ mass}}} - \underbrace{\frac{e^2}{(\underset{\sim}{x}^2 + \underset{\sim}{y}^2 + \underset{\sim}{z}^2)^{1/2}}}_{\substack{\text{coulomb} \\ \text{attraction;} \\ \underset{\sim}{x}, \underset{\sim}{y}, \underset{\sim}{z} \text{ are} \\ \text{coordinate of } e^- \\ \text{w/r to proton.}}}$$

$$\langle \underset{\sim}{H} \rangle = \frac{\langle p_x^2 \rangle + \langle p_y^2 \rangle + \langle p_z^2 \rangle}{2m} - e^2 \left\langle \frac{1}{(\underset{\sim}{x}^2 + \underset{\sim}{y}^2 + \underset{\sim}{z}^2)^{1/2}} \right\rangle$$

seek to <u>minimize</u> $\langle \underset{\sim}{H} \rangle$; that will be the ground state. We'll find that the closer you squish the electron into the proton, thereby reducing the potential energy contribution, the ensuing localization pushes the kinetic energy up (via uncertainty principle). Lowest energy is result of a compromise between the two terms.

To make progress, need some relationships:

$$\langle \underset{\sim}{p}_x^2 \rangle = \langle p_x \rangle^2 + (\Delta p_x)^2$$

variance

recall: $(\Delta p_x)^2 = \langle \psi | (\underset{\sim}{p}_x - \langle \underset{\sim}{p}_x \rangle)^2 | \psi \rangle$

$$= \langle \psi | \underset{\sim}{p}_x^2 | \psi \rangle - 2 \langle \psi | \underset{\sim}{p}_x | \psi \rangle \langle \underset{\sim}{p}_x \rangle + \langle \psi | \langle \underset{\sim}{p}_x \rangle^2 | \psi \rangle$$

$$= \langle \underset{\sim}{p}_x^2 \rangle - 2 \langle \underset{\sim}{p}_x \rangle^2 + \langle \underset{\sim}{p}_x \rangle^2 \cancel{\langle \psi | \psi \rangle}^{1}$$

$$(\Delta p_x)^2 = \langle \underset{\sim}{p}_x^2 \rangle - \langle \underset{\sim}{p}_x \rangle^2$$

$$\langle \underset{\sim}{p}_x^2 \rangle = \langle \underset{\sim}{p}_x \rangle^2 + (\Delta p_x)^2$$ same for y, z

$\Rightarrow$ minimizing $\underset{\sim}{H}$ suggests $\langle p_x \rangle = \langle p_y \rangle = \langle p_z \rangle = 0$

then

$$\langle \underset{\sim}{H} \rangle = \frac{(\Delta p_x)^2 + (\Delta p_y)^2 + (\Delta p_z)^2}{2m} - e^2 \left\langle \frac{1}{(\underset{\sim}{X}^2 + \underset{\sim}{Y}^2 + \underset{\sim}{Z}^2)^{1/2}} \right\rangle$$

"Handwaving"

$$\left\langle \frac{1}{(\underset{\sim}{X}^2 + \underset{\sim}{Y}^2 + \underset{\sim}{Z}^2)} \right\rangle^{1/2} \simeq \frac{1}{\langle (\underset{\sim}{X}^2 + \underset{\sim}{Y}^2 + \underset{\sim}{Z}^2)^{1/2} \rangle} \simeq \frac{1}{(\langle \underset{\sim}{X}^2 \rangle + \langle \underset{\sim}{Y}^2 \rangle + \langle \underset{\sim}{Z}^2 \rangle)^{1/2}}$$

$$\langle \underset{\sim}{x}^2 \rangle = \langle \underset{\sim}{x} \rangle^2 + (\Delta x)^2 \quad \text{et cetera}$$

set to zero, since origin is arbitrary!

$$\langle \underset{\sim}{H} \rangle \simeq \frac{(\Delta p_x)^2 + (\Delta p_y)^2 + (\Delta p_z)^2}{2m} - \frac{e^2}{\left[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\right]^{1/2}}$$

want to minimize ... most minimal

$$\left.\begin{array}{l}(\Delta p_x)^2 = (\Delta p_y)^2 = (\Delta p_z)^2 \\ (\Delta x)^2 = (\Delta y)^2 = (\Delta z)^2\end{array}\right\} \text{True, but not proven here}$$

$$\langle \underset{\sim}{H} \rangle \cong \frac{3(\Delta p_x)^2}{2m} - \frac{e^2}{\sqrt{3}\,\Delta x}$$

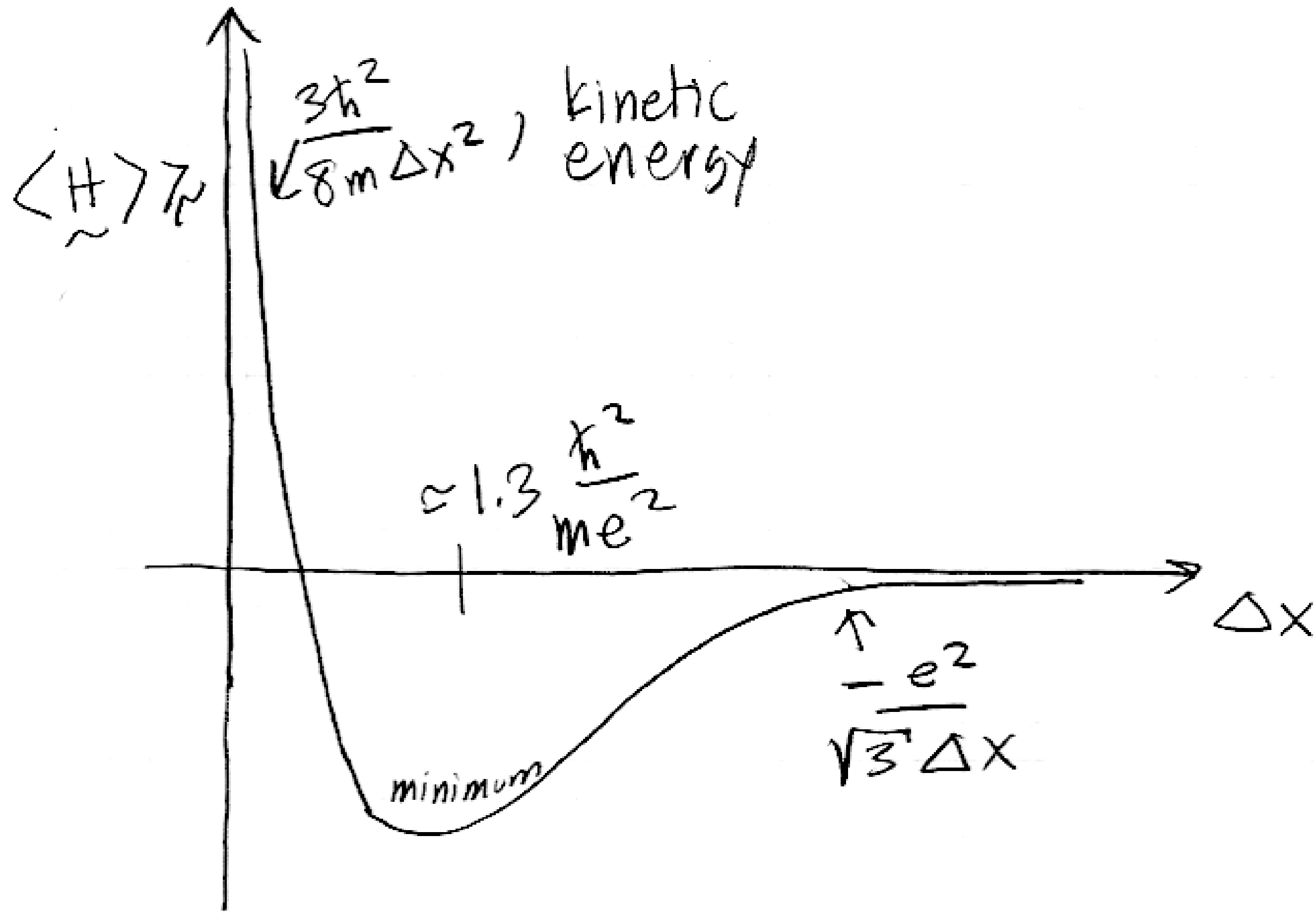
$$\Delta p_x \Delta x > \frac{\hbar}{2}$$

$$(\Delta p_x)^2 > \left(\frac{\hbar}{2\Delta x}\right)^2$$

$$\langle \underset{\sim}{H} \rangle \geq \frac{3\hbar^2}{8m\Delta x^2} - \frac{e^2}{\sqrt{3}\,\Delta x}$$

kinetic — dominant as $\Delta x \to 0$

potential — dominant as $\Delta x \to \infty$



$$\frac{d\langle \underset{\sim}{H} \rangle}{d(\Delta x)} \gtrsim -\frac{3}{4}\frac{\hbar^2}{m(\Delta x)^3} + \frac{e^2}{\sqrt{3}(\Delta x)^2} = 0$$

$$\frac{1}{(\Delta x)} = \frac{4}{3\sqrt{3}}\frac{me^2}{\hbar^2}$$

$$\Delta x = \frac{3\sqrt{3}}{4}\frac{\hbar^2}{me^2} \simeq 1.3\frac{\hbar^2}{me^2}$$

$$\langle \underset{\sim}{H} \rangle \gtrsim \frac{\cancel{3}\hbar^2}{\cancel{8}m\,\frac{9\cdot\cancel{3}}{\cancel{16}_2}\frac{\hbar^4}{m^2e^4}} - \frac{e^2}{\sqrt{3}\cdot\frac{3\cdot\sqrt{3}}{4}\frac{\hbar^2}{me^2}}$$

$$\gtrsim \frac{2}{9}\cdot\frac{me^4}{\hbar^2} - \frac{4}{9}\frac{me^4}{\hbar^2}$$

$$\boxed{\langle \underset{\sim}{H} \rangle \gtrsim -\frac{2}{9}\frac{me^4}{\hbar^2}} \quad \text{exact} \quad -\frac{1}{2}\frac{me^4}{\hbar^2}$$

Product Space

particle #1 $\rightarrow |\psi\rangle$ state vector

particle #2 $\rightarrow |\phi\rangle$ state vector

$|\psi\rangle|\phi\rangle$ means: measure #1, get info described by $|\psi\rangle$

measure #2, get info described by $|\phi\rangle$

Not all states in a product space are products! Key is linear superposition

#1 $\rightarrow |\psi_a\rangle \quad |\psi_b\rangle$ are 2 possible states of #1

#2 $\rightarrow |\phi_a\rangle \quad |\phi_b\rangle$

then: $|\psi_a\rangle|\phi_a\rangle + |\psi_b\rangle|\phi_b\rangle$

is in the product space, but is not in a product. It is "entangled," though.

"Example"

$|x_1\rangle$: $\underset{\sim}{X}_1 |x_1\rangle = x_1 |x_1\rangle$ (eigenstate of particle #1's position)

$|x_2\rangle$: $\underset{\sim}{X}_2 |x_2\rangle = x_2 |x_2\rangle$

idea : $|x_1\rangle|x_2\rangle$ is in the product space

$$\underset{\sim}{X}_1 \left[|x_1\rangle|x_2\rangle\right] = x_1 |x_1\rangle|x_2\rangle$$

$$\underset{\sim}{X}_2 \left[|x_1\rangle|x_2\rangle\right] = x_2 |x_1\rangle|x_2\rangle$$

$|x_1\rangle|x_2\rangle$: $|x_1\rangle$ is in one space (#1)
$|x_2\rangle$ is in another (#2)

$\underset{\sim}{X}_1$: in #1's space, $\underset{\sim}{X}_1$
in #2's space, $\underset{\sim}{1}$

also : $\underset{\sim}{X}_1^{(1)\otimes(2)} = \underset{\sim}{X}_1^{(1)} \otimes \underset{\sim}{1}^{(2)}$ (formally)

$\otimes$ means paste together

$$|x_1\rangle \otimes |x_2\rangle = |x_1\rangle|x_2\rangle \text{ or } |x_1 x_2\rangle$$

Physics

$$\frac{1}{\sqrt{2}}\left[|x_1' x_2'\rangle + |x_1'' x_2''\rangle\right] \qquad x_1' \neq x_1'', \quad x_2' \neq x_2''$$

make wavefunction: project into eigenbras of position:

$$\frac{1}{\sqrt{2}} \langle x_1 x_2 | \left[|x_1' x_2'\rangle + |x_1'' x_2''\rangle \right] = \text{(distributes)}$$

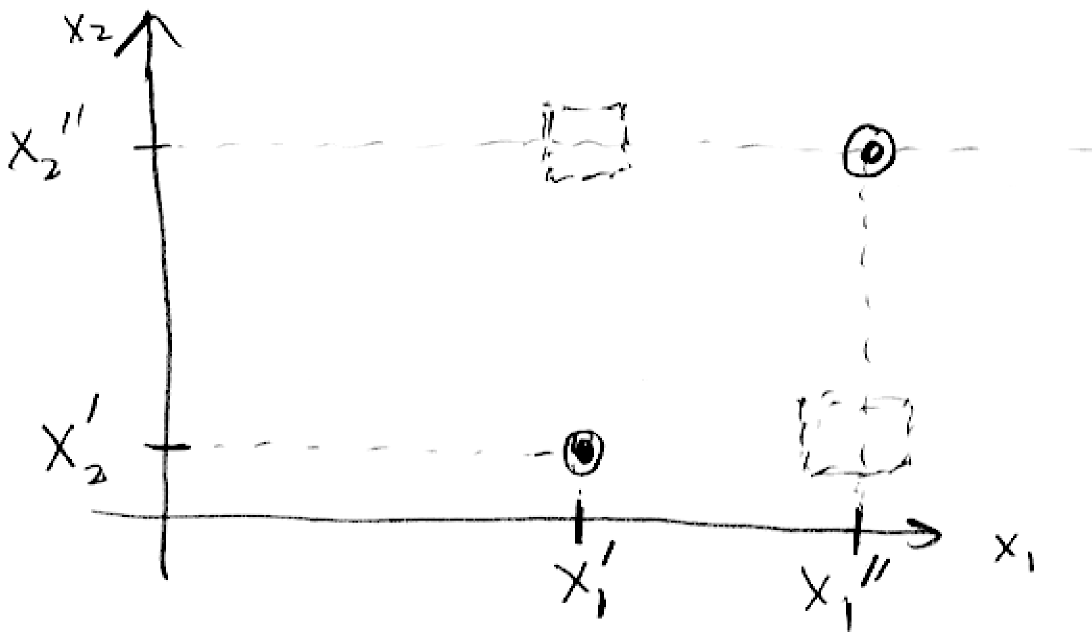
$$= \frac{1}{\sqrt{2}} \langle x_1 x_2 | x_1' x_2' \rangle + \frac{1}{\sqrt{2}} \langle x_1 x_2 | x_1'' x_2'' \rangle$$

x_2 passes through x_1' but not x_2', etc.

$$= \frac{1}{\sqrt{2}} \underbrace{\langle x_1 | x_1' \rangle}_{\delta(x_1 - x_1')} \underbrace{\langle x_2 | x_2' \rangle}_{\delta(x_2 - x_2')} + \frac{1}{\sqrt{2}} \underbrace{\langle x_1 | x_1'' \rangle}_{\delta(x_1 - x_1'')} \underbrace{\langle x_2 | x_2'' \rangle}_{\delta(x_2 - x_2'')}$$

Wavefunction:

$$= \frac{1}{\sqrt{2}} \delta(x_1 - x_1')\delta(x_2 - x_2') + \frac{1}{\sqrt{2}} \delta(x_1 - x_1'')\delta(x_2 - x_2'')$$



Measure $x_1 \rightarrow$ get $x_1' \Rightarrow$ know $x_2 = x_2'$ without measurement (never see x_1', x_2'').

Measure $x_1 \rightarrow$ get $x_1'' \Rightarrow$ know $x_2 = x_2''$ without measurement (never see x_1'', x_2')

"Entanglement", non-product state in product space.

Compare:

$$\left[\frac{1}{\sqrt{2}}\left(|x_1'\rangle + e^{i\theta_1}|x_1''\rangle\right)\right]\left[\frac{1}{\sqrt{2}}\left(|x_2'\rangle + e^{i\theta_2}|x_2''\rangle\right)\right]$$

θ_1, θ_2 = real #'s

this is a product state, in product space.

Matrices That Represent Product Operators

Best to work through example:
Exercise 10.1.2 p. 257:

Particle #1: two states, call eigenstates
$|+\rangle, |-\rangle$

Particle #2: two states in different space, also call
$|+\rangle, |-\rangle$ (just mentally know, for #2)

4 product states span the product space

$|+\rangle|+\rangle$ or $|+\rangle \otimes |+\rangle$ or $|++\rangle$
$|+\rangle|-\rangle$ or $|+\rangle \otimes |-\rangle$ or $|+-\rangle$
$|-\rangle|+\rangle$ or $|-\rangle \otimes |+\rangle$ or $|-+\rangle$
$|-\rangle|-\rangle$ or $|-\rangle \otimes |-\rangle$ or $|--\rangle$