Back to Coordinate Space

key equation: $\underset{\sim}{a}|0\rangle = 0$

$\varepsilon = 0 + \frac{1}{2} = \frac{1}{2}$

$E = \frac{1}{2}\hbar\omega$

$$\underset{\sim}{a} = \frac{1}{\sqrt{2}}\left(\frac{\underset{\sim}{x}}{b} + \frac{ib}{\hbar}\underset{\sim}{p}\right)$$

represent in coordinate space (could do momentum space too)

$\underset{\sim}{x} \doteq x$

$\underset{\sim}{p} \doteq \frac{\hbar}{i}\frac{d}{dx}$

$|0\rangle \doteq \langle x|0\rangle \equiv \psi_0(x)$

mom

$$\begin{cases} \underset{\sim}{x} \doteq i\hbar\frac{d}{dp} \\ \underset{\sim}{p} \doteq p \\ |0\rangle \doteq \langle p|0\rangle \equiv \psi_0(p) \end{cases}$$

$$\frac{1}{\sqrt{2}}\left(\frac{x}{b} + \frac{ib}{\hbar}\frac{\hbar}{i}\frac{d}{dx}\right)\psi_0(x) = 0$$

let $y = \frac{x}{b}$ or $x = by$

$$\frac{1}{\sqrt{2}}\left(y + \frac{d}{dy}\right)\psi_0(y) = 0$$

$$y\psi_0 = -\frac{d\psi_0}{dy}$$

$$\text{or } -y\,dy = \frac{d\psi_0}{\psi_0}$$

$$-\frac{1}{2}y^2 = \ln\psi_0$$

$$\psi_0 = Ae^{-\frac{1}{2}y^2}$$

want $\int_{-\infty}^{\infty} dx\,|A|^2 e^{-\frac{x^2}{b^2}} = 1$ for normalization or $b|A|^2 \underbrace{\int_{-\infty}^{\infty} \frac{dx}{b} e^{-\frac{x^2}{b^2}}}_{y = x/b} = 1$

$$b|A|^2 \underbrace{\int_{-\infty}^{\infty} dy\, e^{-y^2}}_{\sqrt{\pi}} = 1 \qquad \left(\begin{matrix}\text{p. } 659 \\ \text{A.2.1}\end{matrix}\right)$$

$$|A|^2 = \frac{1}{b\sqrt{\pi}} \qquad b = \sqrt{\frac{\hbar}{m\omega}}$$

$$|A|^2 = \left(\frac{m\omega}{\hbar\pi}\right)^{1/2}$$

$$A = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} e^{i\phi} \qquad \begin{matrix}\phi = \text{real} \\ \text{choose } \phi = 0\end{matrix}$$

$$\boxed{\begin{aligned}\psi_0(x) &= \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} e^{-\frac{1}{2}\frac{x^2}{b^2}} \\ &= \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} e^{-\frac{1}{2}y^2}\end{aligned}}$$

$|1\rangle$? $\quad \underset{\sim}{a}^{+}|0\rangle = \sqrt{1}\,|1\rangle = |1\rangle$

$$\underset{\sim}{a}^{+} = \frac{1}{\sqrt{2}}\left(\frac{x}{b} - \frac{ib}{\hbar}p\right) \doteq \frac{1}{\sqrt{2}}\left(y - \frac{d}{dy}\right)$$

so, $\psi_1 = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} \frac{1}{\sqrt{2}}\left(y - \frac{d}{dy}\right) e^{-\frac{1}{2}y^2}$

$$\boxed{\psi_1(y) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} \frac{1}{\sqrt{2}}(y - -y)\, e^{-\frac{1}{2}y^2} = \left(\frac{m\omega}{\hbar\pi}\right)^{1/2} \sqrt{2}\, y\, e^{-\frac{1}{2}y^2}}$$

can extend to higher n.

$$|n\rangle = \frac{a^\dagger}{\sqrt{n}} \text{ (n times)} \cdots \frac{a^\dagger}{\sqrt{3}}\frac{a^\dagger}{\sqrt{2}}\frac{a^\dagger}{\sqrt{1}}|0\rangle$$

$$= \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$$

so, $$\psi_n(y) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}\left(y - \frac{d}{dy}\right)^n e^{-\frac{1}{2}y^2}$$

7.3.22 p.195 $$\psi_n(y) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4}\frac{1}{\sqrt{2^n n!}} e^{-\frac{1}{2}y^2} H_n(y)$$

and so: $$\boxed{H_n(y) = e^{+\frac{1}{2}y^2}\left(y - \frac{d}{dy}\right)e^{-\frac{1}{2}y^2}}$$

Chapter 9 (skip Chapter 8)

generalize $[X, P] = i\hbar \longrightarrow$ antihermitian $(i\hbar)^* = -i\hbar$

with Ω = Hermitian and Λ = Hermitian

consider $$(\Omega\Lambda - \Lambda\Omega)^\dagger = \Lambda^\dagger\Omega^\dagger - \Omega^\dagger\Lambda^\dagger = \Lambda\Omega - \Omega\Lambda$$
$$= -(\Omega\Lambda - \Lambda\Omega)$$

or $[\Omega, \Lambda]^\dagger = -[\Omega, \Lambda]$ antiHermitian

make $[\Omega, \Lambda] = i\Gamma$ where Γ = Hermitian

$$(\Delta\Omega)^2(\Delta\Lambda)^2 \equiv \langle\psi|(\underset{\sim}{\Omega}-\langle\underset{\sim}{\Omega}\rangle)^2|\psi\rangle\langle\psi|(\underset{\sim}{\Lambda}-\langle\underset{\sim}{\Lambda}\rangle)^2|\psi\rangle$$

where $\langle\underset{\sim}{\Omega}\rangle = \langle\psi|\underset{\sim}{\Omega}|\psi\rangle$ is a c-number

$\langle\underset{\sim}{\Lambda}\rangle = \langle\psi|\underset{\sim}{\Lambda}|\psi\rangle$ is also a c-number.

Redefine $\hat{\underset{\sim}{\Omega}} \equiv \underset{\sim}{\Omega} - \langle\underset{\sim}{\Omega}\rangle$

$\hat{\underset{\sim}{\Lambda}} \equiv \underset{\sim}{\Lambda} - \langle\underset{\sim}{\Lambda}\rangle$

c-number (commuting number)

$$[\hat{\underset{\sim}{\Omega}},\hat{\underset{\sim}{\Lambda}}] = [\underset{\sim}{\Omega}-\langle\underset{\sim}{\Omega}\rangle, \underset{\sim}{\Lambda}-\langle\underset{\sim}{\Lambda}\rangle]$$

$$= [\underset{\sim}{\Omega},\underset{\sim}{\Lambda}] - \underbrace{[\underset{\sim}{\Omega},\langle\underset{\sim}{\Lambda}\rangle] - [\langle\underset{\sim}{\Omega}\rangle,\underset{\sim}{\Lambda}] + [\langle\underset{\sim}{\Omega}\rangle,\langle\underset{\sim}{\Lambda}\rangle]}_{\text{all three are zero}}$$

and $\hat{\underset{\sim}{\Omega}} = \hat{\underset{\sim}{\Omega}}^+$, $\hat{\underset{\sim}{\Lambda}} = \hat{\underset{\sim}{\Lambda}}^+$

$$(\Delta\Omega)^2(\Delta\Lambda)^2 = \langle\psi|\hat{\underset{\sim}{\Omega}}^2|\psi\rangle\langle\psi|\hat{\underset{\sim}{\Lambda}}^2|\psi\rangle$$

$$= \langle\psi|\hat{\underset{\sim}{\Omega}}^+\hat{\underset{\sim}{\Omega}}|\psi\rangle\langle\psi|\hat{\underset{\sim}{\Lambda}}^+\hat{\underset{\sim}{\Lambda}}|\psi\rangle$$

call $|\hat{\underset{\sim}{\Lambda}}\psi\rangle = \hat{\underset{\sim}{\Lambda}}|\psi\rangle$ $\qquad |\hat{\underset{\sim}{\Omega}}\psi\rangle = \hat{\underset{\sim}{\Omega}}|\psi\rangle$

then $\langle\hat{\underset{\sim}{\Lambda}}\psi| = \langle\psi|\underset{\sim}{\Lambda}^+$ $\qquad \langle\hat{\underset{\sim}{\Omega}}\psi| = \langle\psi|\hat{\underset{\sim}{\Omega}}^+$

so

$$(\Delta\Omega)^2(\Delta\Lambda)^2 = \langle\hat{\underset{\sim}{\Omega}}\psi|\hat{\underset{\sim}{\Omega}}\psi\rangle\langle\hat{\underset{\sim}{\Lambda}}\psi|\hat{\underset{\sim}{\Lambda}}\psi\rangle$$

now use Schwarz inequality: p. 17 1.3.19

$$\langle V|V\rangle\langle W|W\rangle \geq |\langle V|W\rangle|^2$$

$$(\Delta\Omega)^2(\Delta\Lambda)^2 \geq |\langle\hat{\underset{\sim}{\Omega}}\psi|\hat{\underset{\sim}{\Lambda}}\psi\rangle|^2$$

$$\langle\hat{\underset{\sim}{\Omega}}\psi|\hat{\underset{\sim}{\Lambda}}\psi\rangle = \big(\langle\psi|\underbrace{\hat{\underset{\sim}{\Omega}}^{+}}_{\text{self adjoint}}\big)\big(\hat{\underset{\sim}{\Lambda}}|\psi\rangle\big) = \langle\psi|\hat{\underset{\sim}{\Omega}}\hat{\underset{\sim}{\Lambda}}|\psi\rangle$$

$$(\Delta\Omega)^2(\Delta\Lambda)^2 \geq |\langle\psi|\hat{\underset{\sim}{\Omega}}\hat{\underset{\sim}{\Lambda}}|\psi\rangle|^2$$

$$\underset{\sim}{\Omega}\underset{\sim}{\Lambda} = \frac{1}{2}\underbrace{(\hat{\underset{\sim}{\Omega}}\hat{\underset{\sim}{\Lambda}}+\hat{\underset{\sim}{\Lambda}}\hat{\underset{\sim}{\Omega}})}_{\text{hermitian}} + \frac{1}{2}\underbrace{(\hat{\underset{\sim}{\Omega}}\hat{\underset{\sim}{\Lambda}}-\hat{\underset{\sim}{\Lambda}}\hat{\underset{\sim}{\Omega}})}_{\text{antihermitian}}$$

$$\hat{\underset{\sim}{\Delta}} \equiv \hat{\underset{\sim}{\Omega}}\hat{\underset{\sim}{\Lambda}}+\hat{\underset{\sim}{\Lambda}}\hat{\underset{\sim}{\Omega}} = [\hat{\underset{\sim}{\Omega}},\hat{\underset{\sim}{\Lambda}}]_+ \qquad i\underset{\sim}{\Gamma},\ \underset{\sim}{\Gamma}^{+}=\underset{\sim}{\Gamma}$$

$\langle\psi|\hat{\underset{\sim}{\Delta}}|\psi\rangle$ = real # $\qquad i\langle\psi|\underset{\sim}{\Gamma}|\psi\rangle$ = imaginary #

$$\langle\psi|\hat{\underset{\sim}{\Omega}}\hat{\underset{\sim}{\Lambda}}|\psi\rangle = \frac{1}{2}\underbrace{\langle\psi|\hat{\underset{\sim}{\Delta}}|\psi\rangle}_{\text{real}} + \underbrace{\frac{i}{2}\underbrace{\langle\psi|\underset{\sim}{\Gamma}|\psi\rangle}_{\text{real}}}_{\text{imaginary}}$$

$$|\langle\psi|\hat{\underset{\sim}{\Omega}}\hat{\underset{\sim}{\Lambda}}|\psi\rangle|^2 = \frac{1}{4}|\langle\psi|\hat{\underset{\sim}{\Delta}}|\psi\rangle|^2 + \frac{1}{4}|\langle\psi|i\underset{\sim}{\Gamma}|\psi\rangle|^2$$

Finally:

$$\boxed{(\Delta\Omega)^2(\Delta\Lambda)^2 \geq \frac{1}{4}|\langle\psi|\hat{\underset{\sim}{\Delta}}|\psi\rangle|^2 + \frac{1}{4}|\langle\psi|i\underset{\sim}{\Gamma}|\psi\rangle|^2 \qquad \hat{\underset{\sim}{\Delta}} = [\hat{\underset{\sim}{\Omega}},\hat{\underset{\sim}{\Lambda}}]_+ \quad i\underset{\sim}{\Gamma} = [\hat{\underset{\sim}{\Omega}},\hat{\underset{\sim}{\Lambda}}] = [\underset{\sim}{\Omega},\underset{\sim}{\Lambda}]}$$

Back to $\underset{\sim}{\Omega} = \underset{\sim}{X} \quad \underset{\sim}{\Lambda} = \underset{\sim}{P}$

$$[\underset{\sim}{X}, \underset{\sim}{P}] = i\hbar$$

$$(\Delta x)^2(\Delta p)^2 \geq \frac{1}{4}|\langle\psi|(\underset{\sim}{X}\underset{\sim}{P} + \underset{\sim}{P}\underset{\sim}{X})|\psi\rangle|^2 + \frac{1}{4}\hbar^2$$

or $$\boxed{\begin{aligned}(\Delta x)^2(\Delta p)^2 &\geq \frac{1}{4}\hbar^2 \\ (\Delta x)(\Delta p) &\geq \frac{1}{2}\hbar\end{aligned}}$$

since $\frac{1}{4}|\langle\psi|(\underset{\sim}{X}\underset{\sim}{P}+\underset{\sim}{P}\underset{\sim}{X})|\psi\rangle|^2 \geq 0$

For <u>equality</u>, two conditions must hold:

1) Schwartz EQUALITY meaning:

$$\underset{\sim}{\Omega}|\psi\rangle = c\,\underset{\sim}{\hat{\Lambda}}|\psi\rangle$$

2) Anti-commutator term must vanish

<u>$\underset{\sim}{X}$ + $\underset{\sim}{P}$</u>

$$(\underset{\sim}{P} - \langle\underset{\sim}{P}\rangle)|\psi\rangle = c(\underset{\sim}{X} - \langle\underset{\sim}{X}\rangle)|\psi\rangle$$

rep in coordinate space.

$$\left(\frac{\hbar}{i}\frac{d}{dx} - \langle p\rangle\right)\psi(x) = c\left(x - \langle\underset{\sim}{X}\rangle\right)\psi(x)$$

let $x' = x - \langle\underset{\sim}{X}\rangle$

then $$\frac{d\psi}{\psi} = \frac{i}{\hbar}(\langle p\rangle + c x')\,dx'$$

$$\ln \Psi = \alpha + \frac{i}{\hbar}\left(\langle p \rangle x' + c\left(\tfrac{1}{2}x'^2\right)\right)$$

$$\boxed{\Psi(x) = A\, e^{\frac{i\langle p\rangle x'}{\hbar}}\, e^{\frac{icx'^2}{2\hbar}} \quad \text{satisfies Schwartz}}$$

at this point, c is unconstrained...
can be any general complex number.
The <u>second</u> condition, that the anticommutator vanish, constrains c.

starting from Schwartz:

$$(\underset{\sim}{p} - \langle \underset{\sim}{p} \rangle)|\Psi\rangle = c\underset{\sim}{x}'|\Psi\rangle$$

$$\underset{\sim}{x}'(\underset{\sim}{p} - \langle \underset{\sim}{p} \rangle)|\Psi\rangle = c\underset{\sim}{x}'^2|\Psi\rangle$$

$$\langle\Psi|\underset{\sim}{x}'(\underset{\sim}{p} - \langle \underset{\sim}{p} \rangle)|\Psi\rangle = c\langle\Psi|\underset{\sim}{x}'^2|\Psi\rangle$$

$$\langle\Psi|\underbrace{(\underset{\sim}{p} - \langle \underset{\sim}{p} \rangle)}_{\text{consider left operation}}\underset{\sim}{x}'|\Psi\rangle = c^*\langle\Psi|\underset{\sim}{x}'^2|\Psi\rangle$$

$$\langle\Psi|\left(\underset{\sim}{x}'(\underset{\sim}{p} - \langle \underset{\sim}{p} \rangle) + (\underset{\sim}{p} - \langle \underset{\sim}{p} \rangle)\underset{\sim}{x}'\right)|\Psi\rangle$$

$$= \underbrace{(c + c^*)}_{\text{must have } c+c^*=0}\underbrace{\langle\Psi|\underset{\sim}{x}'^2|\Psi\rangle}_{\geq 0 \text{ usually } > 0} = 0$$

must have $c + c^* = 0$ $\boxed{c = \text{pure imaginary}}$