The Oscillator in the Energy Basis

$[\underset{\sim}{X}, \underset{\sim}{P}] = i\hbar$ in <u>any</u> basis p. 203

consider operator.

i) (imperfect) $\underset{\sim}{X} + i\underset{\sim}{P}$

(a) why? 1) "simpler solution to QSHO follows"

2) $\underset{\sim}{H}$(QSHO) $\propto$ (sort of) $\underset{\sim}{X}^2 + \underset{\sim}{P}^2$

$\propto (\underset{\sim}{X} + i\underset{\sim}{P})(\underset{\sim}{X} - i\underset{\sim}{P})$ (almost not quite/ (commutator))

3) "factorizes" $\underset{\sim}{H}$

ii) actually, must look out for "scaling factors"!...

recall $\underset{\sim}{H} = \frac{1}{2m} \underset{\sim}{P}^2 + \frac{1}{2} \overset{= m\omega^2}{k} \underset{\sim}{X}^2$ (really)

how do we get these factors right... well, the <u>scale</u> $b = \left(\frac{\hbar}{m\omega}\right)^{1/2}$ was already set up to address these factors

iii) try $\underset{\sim}{a} \equiv \frac{1}{\sqrt{2}} \left(\frac{\underset{\sim}{X}}{b} + \frac{ib}{\hbar} \underset{\sim}{P} \right)$

useful normalization factor

since $\underset{\sim}{P} \doteq \frac{\hbar}{i} \frac{d}{dx}$

(cancel $\hbar$, i, $\frac{1}{\text{length}}$)

then $\underset{\sim}{a}^{+} = \frac{1}{\sqrt{2}}\left(\frac{\underset{\sim}{x}^{+}}{b^{*}} + \frac{i^{*}b^{*}}{\hbar}p^{+}\right)$

$\underset{\sim}{x} = \underset{\sim}{x}^{+}$, $p = p^{+}$ (these must be Hermitian)

$b^{*} = b$, $i^{*} = -i$

$$\underset{\sim}{a}^{+} = \frac{1}{\sqrt{2}}\left(\frac{\underset{\sim}{x}}{b} - \frac{ib}{\hbar}p\right) \quad \text{(got the - sign)}$$

Look at:

$$\underset{\sim}{a}^{+}\underset{\sim}{a} = \frac{1}{2}\left(\frac{\underset{\sim}{x}}{b} - \frac{ib}{\hbar}p\right)\left(\frac{\underset{\sim}{x}}{b} + \frac{ib}{\hbar}p\right)$$

$$= \frac{1}{2}\left\{\frac{\underset{\sim}{x}^{2}}{b^{2}} + \frac{i}{\hbar}(\underset{\sim}{x}p - p\underset{\sim}{x}) + \frac{b^{2}}{\hbar^{2}}p^{2}\right\}$$

$$= \frac{1}{2}\left\{\frac{m\omega}{\hbar}\underset{\sim}{x}^{2} + \frac{\hbar}{\hbar^{2}m\omega}p^{2}\right\} + \frac{i}{2\hbar}i\hbar$$

$$= \frac{1}{\hbar\omega}\cdot\underbrace{\frac{1}{2}\left\{m\omega^{2}x^{2} + \frac{1}{m}p^{2}\right\}}_{\underset{\sim}{H}} - \frac{1}{2}$$

just a number (like adding a constant)

$$\underset{\sim}{a}^{+}\underset{\sim}{a} = \frac{\underset{\sim}{H}}{\hbar\omega} - \frac{1}{2}$$

dimensionless energy $\equiv \underset{\sim}{\hat{H}}$

so, $\boxed{\underset{\sim}{\hat{H}} = \underset{\sim}{a}^{+}\underset{\sim}{a} + \frac{1}{2}}$ "factorization of $\underset{\sim}{H}$"

$\underset{\sim}{H} \neq \underset{\sim}{a}\underset{\sim}{a}^{+} + \frac{1}{2}$ why?

$\underset{\sim}{a}\underset{\sim}{a}^{+} \neq \underset{\sim}{a}^{+}\underset{\sim}{a}$

commutator $[\underset{\sim}{a}, \underset{\sim}{a}^{+}]$

$$= \underset{\sim}{a}\underset{\sim}{a}^{+} - \underset{\sim}{a}^{+}\underset{\sim}{a}$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\underset{\sim}{x}}{b} + \frac{ib}{\hbar}p\right)\frac{1}{\sqrt{2}}\left(\frac{\underset{\sim}{x}}{b} - \frac{ib}{\hbar}p\right) - \frac{1}{\sqrt{2}}\left(\frac{\underset{\sim}{x}}{b} - \frac{ib}{\hbar}p\right)\frac{1}{\sqrt{2}}\left(\frac{\underset{\sim}{x}}{b} + \frac{ib}{\hbar}p\right)$$

cancels

(similar for p^2 term)

$$= \frac{1}{2}\left\{\frac{i}{\hbar}\underbrace{(p\underset{\sim}{x} - \underset{\sim}{x}p)}_{-[\underset{\sim}{x}, p] = -i\hbar} - \frac{i}{\hbar}\underbrace{(\underset{\sim}{x}p - p\underset{\sim}{x})}_{[\underset{\sim}{x}, p] = i\hbar}\right\}$$

$$= \frac{1}{2}\cdot\frac{i}{\hbar}\left[-2i\hbar\right] = +1$$

so,
$$[\underset{\sim}{a}, \underset{\sim}{a}^{+}] = 1$$

$$\underset{\sim}{a}\underset{\sim}{a}^{+} - \underset{\sim}{a}^{+}\underset{\sim}{a} = 1$$

$$\underset{\sim}{a}\underset{\sim}{a}^{+} = \underset{\sim}{a}^{+}\underset{\sim}{a} + 1$$

$$\underset{\sim}{a}^{+}\underset{\sim}{a} = \underset{\sim}{a}\underset{\sim}{a}^{+} - 1$$

$$\boxed{\underset{\sim}{H} = \underset{\sim}{a}^{+}\underset{\sim}{a} + \frac{1}{2} = \underset{\sim}{a}\underset{\sim}{a}^{+} - \frac{1}{2}}$$

The factorization allows a pretty easy solution of QSHO... How

observe: $[\underset{\sim}{a}, \hat{\underset{\sim}{H}}] = [\underset{\sim}{a}, \underset{\sim}{a}^{+}\underset{\sim}{a} - \frac{1}{2}] = [\underset{\sim}{a}, \underset{\sim}{a}^{+}\underset{\sim}{a}]$

you'll see..

$$[\underset{\sim}{a}, \underset{\sim}{a}^{+}\underset{\sim}{a}] = \overbrace{[\underset{\sim}{a}, \underset{\sim}{a}^{+}]}^{1}\underset{\sim}{a} + \underset{\sim}{a}^{+}\overbrace{[\underset{\sim}{a}, \underset{\sim}{a}]}^{0} = \underset{\sim}{a}$$

$$[\underset{\sim}{A}, \underset{\sim}{B}\underset{\sim}{C}] = [\underset{\sim}{A}, \underset{\sim}{B}]\underset{\sim}{C} + \underset{\sim}{B}[\underset{\sim}{A}, \underset{\sim}{C}]$$

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so $[\underset{\sim}{a}, \underset{\sim}{H}] = \underset{\sim}{a}$ $\left(\text{also, } [\underset{\sim}{a}^{+}, \underset{\sim}{H}] = -\underset{\sim}{a}^{+}\right)$.

$\underset{\sim}{a}\underset{\sim}{H} - \underset{\sim}{H}\underset{\sim}{a} = \underset{\sim}{a}$

Suppose we have one eigenstate:

$$\hat{\underset{\sim}{H}}|\varepsilon\rangle = \varepsilon|\varepsilon\rangle$$

$$\underset{\sim}{a}\underset{\sim}{H}|\varepsilon\rangle = \underset{\sim}{a}\,\varepsilon|\varepsilon\rangle = \varepsilon(\underset{\sim}{a}|\varepsilon\rangle)$$

$$= (\underset{\sim}{H}\underset{\sim}{a} + \underset{\sim}{a})|\varepsilon\rangle$$

so $\underset{\sim}{H}\underbrace{\underset{\sim}{a}|\varepsilon\rangle} = (\varepsilon - 1)\underbrace{\underset{\sim}{a}|\varepsilon\rangle}$.

We got another eigenstate, namely $\underset{\sim}{a}|\varepsilon\rangle$, with eigenvalue $(\varepsilon-1)$!

$$\underset{\sim}{a}|\varepsilon\rangle = C_\varepsilon|\varepsilon - 1\rangle$$

normalization constant

$$\underset{\sim}{a}^{+}\underset{\sim}{H}\,|\varepsilon\rangle = \varepsilon\left(\underset{\sim}{a}^{+}|\varepsilon\rangle\right)$$
$$= \left(\underset{\sim}{H}\underset{\sim}{a}^{+} - \underset{\sim}{a}^{+}\right)|\varepsilon\rangle$$
$$\underset{\sim}{H}\left(\underset{\sim}{a}^{+}|\varepsilon\rangle\right) = (\varepsilon+1)\,\underset{\sim}{a}^{+}|\varepsilon\rangle$$

Yet another eigenstate, now with eigenvalue $(\varepsilon+1)$!

$$\underset{\sim}{a}^{+}|\varepsilon\rangle = c_{\varepsilon+1}\,|\varepsilon+1\rangle$$

$\underset{\sim}{a} \rightarrow$ called lowering operator or destruction operator.

$\underset{\sim}{a}^{+} \rightarrow$ raising operator creation operator.

Eigenvalues of $\underset{\sim}{\hat{H}}$:

lower raise

$$\varepsilon-4,\ \varepsilon-3,\ \varepsilon-2,\ \varepsilon-1,\ \varepsilon,\ \varepsilon+1,\ \varepsilon+2$$

however, $\langle\varepsilon'|\underset{\sim}{\hat{H}}|\varepsilon'\rangle = \varepsilon'\langle\varepsilon'|\varepsilon'\rangle = \varepsilon'$

$$= \varepsilon' \geq 0$$

$\therefore$ must be that the "chain" of eigenvalues terminates. To do so:

$$\underset{\sim}{a}\,|\bar{\varepsilon}\rangle = 0$$
$$\underset{\sim}{a}^{+}\underset{\sim}{a}\,|\bar{\varepsilon}\rangle = 0$$
$$\left(\underset{\sim}{\hat{H}} - \tfrac{1}{2}\right)|\bar{\varepsilon}\rangle = 0 \Rightarrow \underset{\sim}{H}|\bar{\varepsilon}\rangle = \tfrac{1}{2}|\bar{\varepsilon}\rangle$$

So, the state with eigenvalue $\bar{\varepsilon} = \frac{1}{2}$ <u>cannot</u> be lowered, because $\underset{\sim}{a}|\frac{1}{2}\rangle = 0$, (not $|-\frac{1}{2}\rangle$).

The eigenvalues of $\underset{\sim}{\hat{H}}$ are:

$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \ldots \ldots n + \frac{1}{2} \quad n = 0, 1, \ldots$

and those of $\underset{\sim}{H} = \hbar\omega \underset{\sim}{\hat{H}}$ are:

$\frac{1}{2}\hbar\omega, \frac{3}{2}\hbar\omega, \ldots \ldots \ldots (n + \frac{1}{2})\hbar\omega.$