1(7.1) Eqtn (7.5) $[A, A^\dagger] = 1$; Eqtn (7.26) $u_n = \frac{1}{\sqrt{n!}} (A^\dagger)^n u_0$

Show $A u_n = \sqrt{n}\, u_{n-1}$

i) True for $n=1$:

$$A u_1 = A\left(\frac{1}{\sqrt{1}} (A^\dagger)^1 u_0\right) = A A^\dagger u_0 = (A^\dagger A + [A, A^\dagger]) u_0 = A^\dagger \underbrace{A u_0}_{0} + u_0 = \sqrt{1}\, u_{1-1} \checkmark$$

ii) Suppose for a particular n, $A u_n = \sqrt{n}\, u_{n-1}$

Then: $A(u_{n+1}) = A\left(\frac{1}{\sqrt{(n+1)!}} (A^\dagger)^{n+1} u_0\right) = A\left(\frac{1}{\sqrt{n+1}} \frac{1}{\sqrt{n!}} A^\dagger (A^\dagger)^n u_0\right)$

$$= \frac{1}{\sqrt{n+1}} A A^\dagger \underbrace{\left(\frac{1}{\sqrt{n!}} (A^\dagger)^n u_0\right)}_{= u_n} = \frac{1}{\sqrt{n+1}} (A^\dagger A + 1) u_n$$

$$= \frac{1}{\sqrt{n+1}} (A^\dagger \underbrace{A u_n}_{= \sqrt{n}\, u_{n-1}} + u_n) = \frac{1}{\sqrt{n+1}} (\sqrt{n}\, A^\dagger u_{n-1} + u_n)$$

$$= \frac{1}{\sqrt{n+1}} \left(\sqrt{n} \underbrace{\left[\frac{1}{\sqrt{(n-1)!}} (A^\dagger)^{n-1} u_0\right]}_{= \sqrt{n}\, u_n} + u_n\right) = \frac{1}{\sqrt{n+1}} (n u_n + u_n)$$

$$= \frac{n+1}{\sqrt{n+1}} u_n = \sqrt{n+1}\, u_n$$

Since its true for $n+1$ if true for n and its true for $n=1$, it holds for all n. ✓

2. (7.3) From Eqtns. (7-4) we see that $x = \sqrt{\frac{\hbar}{2m\omega}} (A + A^\dagger)$

From prob 1. we have $A|u_n\rangle = \sqrt{n}\, |u_{n-1}\rangle$ (1)

$$\Rightarrow \langle u_n | A^\dagger = \sqrt{n}\, \langle u_{n-1}|$$ (2)

$$\langle u_n | x | u_m \rangle = \langle u_n | \left(\sqrt{\frac{\hbar}{2m\omega}} (A + A^\dagger)\right) | u_m \rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \left\{ \langle u_n | \underbrace{A | u_m \rangle}_{\sqrt{m}\, |u_{m-1}\rangle} + \underbrace{\langle u_n | A^\dagger | u_m \rangle}_{\sqrt{n}\, \langle u_{n-1}|} \right\} \quad \text{by (1), (2)}$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \left\{ \sqrt{m} \underbrace{\langle u_n | u_{m-1} \rangle}_{\delta_{n,m-1}} + \sqrt{n} \underbrace{\langle u_{n-1} | u_m \rangle}_{\delta_{n-1,m}} \right\}$$

$$\boxed{\langle u_n | x | u_m \rangle = \sqrt{\frac{\hbar}{2m\omega}} \left\{ \sqrt{n+1}\, \delta_{n,m-1} + \sqrt{n}\, \delta_{n,m+1} \right\}}$$

which is clearly zero unless $n = m \pm 1$