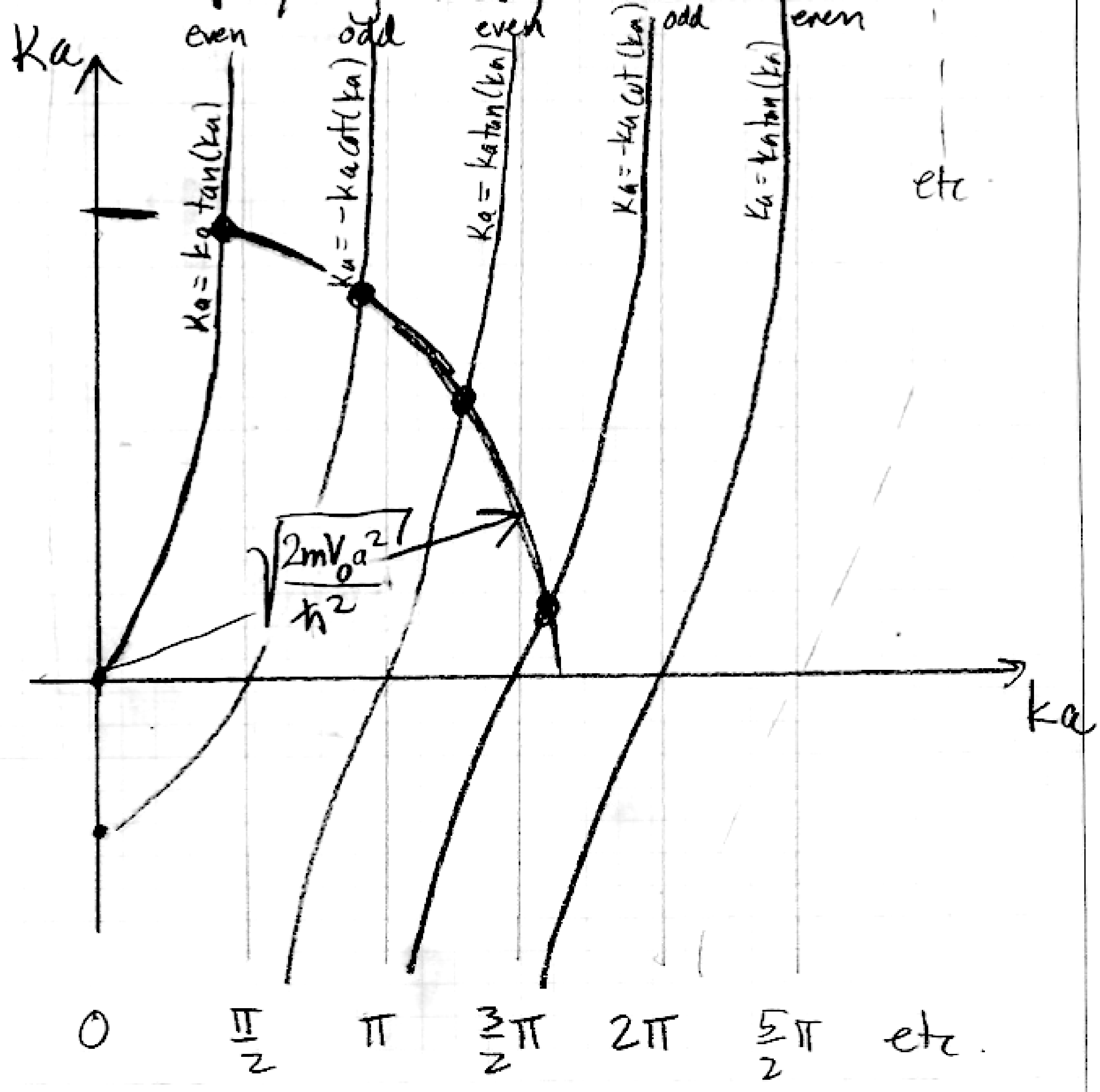
1. Refer to p. 164, Exercise 5.2.6.

even solutions: $(ka)\tan(ka) = Ka$

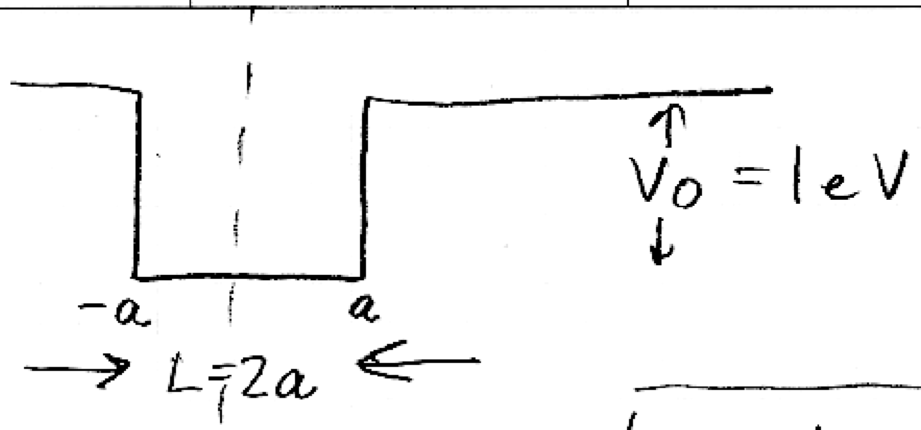
odd solutions: $-(ka)\cot(ka) = Ka$

all solutions: $(ka)^2 + (Ka)^2 = \frac{2mV_0 a^2}{\hbar^2}$

make a diagram with horizontal axis displaying ka, the vertical Ka



(a)



So, $L = 1\,nm = 2a$, $\boxed{a = \frac{1}{2}\,nm}$

Look at the diagram on the previous page. The curves resulting from $Ka = (ka)\tan(ka)$ and $Ka = -(ka)\cot(ka)$ are the same for every square well problem. What changes is the radius of the circle; that radius is equal to $\sqrt{\frac{2mV_0 a^2}{\hbar^2}}$. Think of the intersections of this circle with the other curves as a function of the radius of the circle. No matter how small $\sqrt{\frac{2mV_0 a^2}{\hbar^2}}$ is, there is always one intersection with the "first branch" of $Ka = (ka)\tan(ka)$, indicating that there is always one even bound state. As the radius increases, when it reaches a value of $\pi/2$, a second (odd) bound state appears. And then when the radius increases to a value of $2 \cdot \pi/2$ a third (even) state appears

So, the number of bound states, N_b, is:

$$N_b = \text{integer just bigger than } \left(\frac{\sqrt{\frac{2mV_0a^2}{\hbar^2}}}{\left(\frac{\pi}{2}\right)} \right)$$

Two Numerical Routes

MKS

$m = 9.1 \cdot 10^{-31}$ kg (electron)

$V_0 = 1\,eV = 1.6 \cdot 10^{-19}$ Joules

$a = \frac{1}{2} \cdot 10^{-9}$ m

$\hbar = 1.05\,10^{-34}$ J-s

$$\sqrt{\frac{2mV_0a^2}{\hbar^2}}$$

$$= \left[\frac{\cancel{2} \cdot 9.1 \cdot 10^{-31} \cdot 1.6 \cdot 10^{-19} \cdot \frac{1}{\cancel{4}\,2} \cdot 10^{-18}}{1.1 \cdot \cancel{10^{-68}}}\right]^{1/2} \quad 10^{-68}$$

$$= \left[\frac{1}{2} \cdot \frac{9.1 \cdot 1.6}{1.1}\right]^{1/2} \left(\simeq \left[\frac{1}{2} \cdot \frac{9 \cdot 4}{3}\right]^{1/2}\right)$$

$$\left(\simeq \sqrt{6}\right)$$

$$= \sqrt{6.6}$$

$$= 2.6$$

"Atomic Units" p. 669

$mc^2 = 0.511 \cdot 10^6\,eV$

$V_0 = 1\,eV$

$a = \frac{1}{2}$ nm

$\hbar c = 1973\,eV\,\text{Å}$

$= 197.3\,eV \cdot nm$

$$= \sqrt{\frac{2mc^2V_0a^2}{\hbar^2c^2}}$$

$$= \sqrt{\frac{2 \cdot 0.511 \cdot 10^6 \cdot 1 \cdot \frac{1}{4}}{(1.97)^2 \cdot 10^4}}$$

$$= \sqrt{\frac{1.02}{16} \cdot 10^2} \quad \left(\simeq \sqrt{6.4}\right)$$

$$= \sqrt{6.6}$$

$$= 2.6$$

$$\frac{\sqrt{\frac{2mV_0a^2}{\hbar^2}}}{\left(\frac{\pi}{2}\right)} = \frac{2.6}{1.5} = 1.6 \Rightarrow \boxed{\text{2 bound states}}$$

(b) I solved the sets of equations:

$$x^2+y^2=6.6 \qquad x^2+y^2=6.6$$
$$y=x\tan x \qquad y=-x\cot x$$

through a combination of guesswork and iteration. I put the spreadsheet up on the course website.

$$x_1=1.12 = k_1 a \qquad x_2=2.15 = k_2 a$$

$$k_1 = \frac{1.12}{a} = 2.24\cdot10^9\,\frac{1}{m} \qquad k_2=\frac{2.15}{a}=4.3\cdot10^9\,\frac{1}{m}$$

$$E_1 = \frac{\hbar^2 k_1^2}{2m} = \frac{(1.05\cdot10^{-34})^2\cdot(2.24\cdot10^9)^2}{2\cdot9.1\cdot10^{-31}} \qquad E_2=\frac{\hbar^2k_2^2}{2m}$$

$$= \frac{(1.05\cdot10^{-34})(4.3\cdot10^9)^2}{2\cdot9.1\cdot10^{31}}$$

$$\underline{E_1 = 3.04\cdot10^{-20}\,J} \qquad \underline{E_2=1.12\cdot10^{-19}\,J}$$

in eV

$$E_1 = \frac{3.04\cdot10^{-20}\,J}{1.6\cdot10^{-19}\,J/eV} \qquad E_2=\frac{1.12\cdot10^{-19}\,J}{1.6\cdot10^{-19}\,J/eV}$$

$$\boxed{E_1 = 0.19\,eV \qquad E_2=0.70\,eV}$$

(c) $E_2-E_1 = 0.70-0.19 = 0.51\,eV$

$$\hbar\omega = E_2-E_1 \;;\; \frac{2\pi}{\lambda}=k=\frac{\omega}{c}=\frac{E_2-E_1}{\hbar c}$$

$$\lambda = \frac{2\pi\hbar c}{E_2-E_1} = \frac{2\pi\cdot197.3\,eV\,nm}{0.51\cdot1000\,\frac{nm}{\mu m}} = \boxed{2.43\,\mu m}$$

2. (5.3.4) p. 167.

$$\psi = A e^{\frac{ipx}{\hbar}} + B e^{-\frac{ipx}{\hbar}}$$

$$\psi^* = A^* e^{-\frac{ipx}{\hbar}} + B^* e^{\frac{ipx}{\hbar}}$$

$$\frac{d\psi}{dx} = \frac{ip}{\hbar}\left(A e^{\frac{ipx}{\hbar}} - B e^{-\frac{ipx}{\hbar}}\right)$$

$$\frac{d\psi^*}{dx} = \frac{ip}{\hbar}\left(-A^* e^{-\frac{ipx}{\hbar}} + B^* e^{\frac{ipx}{\hbar}}\right)$$

$$\psi^* \frac{d\psi}{dx} - \psi \frac{d\psi^*}{dx}$$

$$= \left(A^* e^{-\frac{ipx}{\hbar}} + B^* e^{\frac{ipx}{\hbar}}\right) \frac{ip}{\hbar}\left(A e^{\frac{ipx}{\hbar}} - B e^{-\frac{ipx}{\hbar}}\right)$$

$$- \left(A e^{\frac{ipx}{\hbar}} + B e^{-\frac{ipx}{\hbar}}\right) \frac{ip}{\hbar}\left(-A^* e^{-ipx} + B^* e^{\frac{ipx}{\hbar}}\right)$$

$$= \frac{ip}{\hbar}\left\{ \begin{array}{l} |A|^2 - \cancel{A^* B e^{-\frac{2ipx}{\hbar}}} + \cancel{B^* A e^{\frac{2ipx}{\hbar}}} - |B|^2 \\ + |A|^2 + \cancel{A^* B e^{-\frac{2ipx}{\hbar}}} - \cancel{B^* A e^{\frac{2ipx}{\hbar}}} - |B|^2 \end{array} \right\}$$

$$= \frac{2ip}{\hbar}\left(|A|^2 - |B|^2\right)$$

$$j = \frac{\hbar}{2mi}\left(\psi^* \frac{d\psi}{dx} - \psi \frac{d\psi^*}{dx}\right) = \frac{\cancel{\hbar}}{2m\cancel{i}} \frac{2\cancel{i}p}{\cancel{\hbar}}\left(|A|^2 - |B|^2\right)$$

$$\boxed{j = \frac{p}{m}\left(|A|^2 - |B|^2\right)}$$

(3) $$\psi(x,0) = e^{\frac{ip_0 x}{\hbar}} \frac{e^{-x^2/2\Delta^2}}{(\pi\Delta^2)^{1/4}}$$

$$\psi^*(x,0) = e^{-\frac{ip_0 x}{\hbar}} \quad ''$$

$$\frac{d\psi}{dx} = \left(\frac{ip_0}{\hbar} - \frac{x}{\Delta^2}\right) e^{\frac{ip_0 x}{\hbar}} \frac{e^{-x^2/2\Delta^2}}{(\pi\Delta^2)^{1/4}}$$

$$\frac{d\psi^*}{dx} = \left(\frac{-ip_0}{\hbar} - \frac{x}{\Delta^2}\right) e^{-\frac{ip_0 x}{\hbar}} \frac{e^{-x^2/2\Delta^2}}{(\pi\Delta^2)^{1/4}}$$

$$\psi^*\frac{d\psi}{dx} - \psi\frac{d\psi^*}{dx} = \left(\frac{ip_0}{\hbar} - \frac{x}{\Delta^2}\right)\frac{e^{-x^2/\Delta^2}}{(\pi\Delta^2)^{1/2}} - \left(\frac{-ip_0}{\hbar} - \frac{x}{\Delta^2}\right)\frac{e^{-x^2/\Delta^2}}{(\pi\Delta^2)^{1/2}}$$

$$= \frac{2ip_0}{\hbar} \frac{e^{-x^2/\Delta^2}}{(\pi\Delta^2)^{1/2}}$$

$$j = \frac{\hbar}{2mi}\left(\psi^*\frac{d\psi}{dx} - \psi\frac{d\psi^*}{dx}\right)$$

$$= \frac{\cancel{\hbar}}{2m\cancel{i}} \cdot \frac{\cancel{2}\cancel{i}p_0}{\cancel{\hbar}} \frac{e^{-x^2/\Delta^2}}{(\pi\Delta^2)^{1/2}}$$

$$\boxed{j = \frac{p_0}{m} \frac{e^{-x^2/\Delta^2}}{(\pi\Delta^2)^{1/2}} = \left(\frac{p_0}{m}\right)|\psi(x,0)|^2}$$

velocity density