# Physics 115A Midterm

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Closed Book; no calculators. For full credit, show your work and make your reasoning clear to the graders.

The 'boldface' notation below is used for operators; thus, $\mathbf{\Omega}$ is an abstract operator. In class we put a 'twiddle' under the $\Omega$ to denote that it was an operator. The symbol $\doteq$ means 'is represented by'.

The quadratic formula for the roots to the equation $ax^2 + bx + c = 0$ is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

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1. (25 pts) Two kets have unit length: $|V_1\rangle$, and $|V_2\rangle$, so $\langle V_1|V_1\rangle = \langle V_2|V_2\rangle = 1$; these two kets are never equal, that is, $|V_1\rangle \neq |V_2\rangle$. The two projection operators are $\mathbf{P_1} = |V_1\rangle\langle V_1|$ and $\mathbf{P_2} = |V_2\rangle\langle V_2|$.
   (a) Suppose $|V_1\rangle$ and $V_2\rangle$ are represented in an orthonormal basis by:
   $$|V_1\rangle \doteq \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad , \quad |V_2\rangle \doteq \begin{bmatrix} -\sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \end{bmatrix}.$$
      i. Find the matrices that represent $\mathbf{P_1}$ and $\mathbf{P_2}$.
      ii. Use the matrix representations to find the matrix that represents the commutator $[\mathbf{P_1}, \mathbf{P_2}]$.
   (b) In *general*, what conditions on $|V_1\rangle$ and $|V_2\rangle$ will guarantee that the commutator $[\mathbf{P_1}, \mathbf{P_2}] = 0$?

2. (20 pts) Consider the linear operator $\mathbf{\Omega}$ which operates on abstract vectors in a space of dimension 2, and which is represented in one particular basis by the matrix:
   $$\mathbf{\Omega} \doteq \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}.$$
   (a) Is $\mathbf{\Omega}$ Hermitian?
   (b) Is $\mathbf{\Omega}$ unitary?
   (c) What are the eigenvalues of $\mathbf{\Omega}$?
   (d) What are the representations of the normalized eigenvectors of $\mathbf{\Omega}$?

**Over...**

3. (40 pts) The linear operators $\mathbf{\Omega}$ and $\Lambda$ operate on abstract vectors in a space of dimension 3, and in one particular orthonormal basis they are represented by the matrix:

$$\mathbf{\Omega} \doteq \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad , \quad \mathbf{\Lambda} \doteq \begin{bmatrix} \frac{2}{3}b & b & \frac{1}{3}b \\ b & b & b \\ \frac{1}{3}b & b & \frac{2}{3}b \end{bmatrix}$$

where $b$ is a non-zero real number.

(a) Do $\mathbf{\Omega}$ and $\mathbf{\Lambda}$ commute?

(b) Is $\mathbf{\Omega}$ unitary?

(c) One eigenket of $\mathbf{\Omega}$, $|\omega_3\rangle$, has an obvious representation in this basis, namely:

$$|\omega_3\rangle \doteq \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix};$$

what is the eigenvalue $\omega_3$ that corresponds to this eigenket?

(d) Find the other two eigenvalues of $\mathbf{\Omega}$; call $\omega_1$ the smaller of the two, and $\omega_2$ the larger of the two.

(e) Find the unitary matrix that transforms the representation of $\mathbf{\Omega}$ given above into the diagonal form:

$$\mathbf{\Omega} \doteq \begin{bmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{bmatrix}$$

(f) Apply the same unitary transformation to $\mathbf{\Lambda}$.

(g) What are the eigenvalues of $\mathbf{\Lambda}$?

4. (15 pts) Numerically evaluate the integral:

$$\int_{-\infty}^{\infty} \delta(4x-2)\,[\frac{1}{2}x^2 - 1]dx$$

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