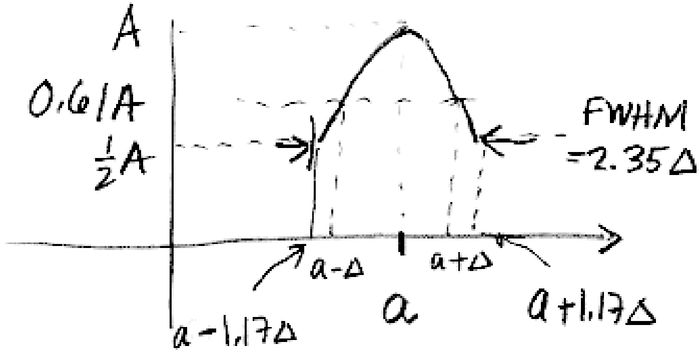
Working With $|x\rangle$, $|p\rangle$ p. 134

$$|\psi\rangle = \int_{-\infty}^{\infty} |x\rangle\, dx\, \underbrace{\langle x|\psi\rangle}_{\psi(x)} = \int_{-\infty}^{\infty} |x\rangle\, dx\, \psi(x)$$

known to exist in x

(one-dimension).

Suppose $\psi(x)$ is a "gaussian" $\psi(x) = A e^{-\frac{(x-a)^2}{2\Delta^2}}$



when $x-a = \pm\Delta$

$x = a \pm \Delta$

$\psi(x) = A e^{-1/2} = 0.61\,A$

also, when

$\psi(x_\pm) = \frac{1}{2}A = A e^{-\frac{(\pm\alpha)^2}{2\Delta^2}}$

$x_+ - a = +\alpha$

$x_- - a = -\alpha$

$2\Delta^2 \ln 2 = \alpha^2$

$\alpha = \sqrt{2}\,\Delta\sqrt{\ln 2}$

$= 1.18\,\Delta$

Gaussian famous as a probability distribution, because it is an important limiting case.

Normalization   (p. 65a)

$$I_0(\alpha) \equiv \int_{-\infty}^{\infty} e^{-\alpha x^2}\, dx = \sqrt{\frac{\pi}{a}}$$

want $\langle\psi|\psi\rangle = 1 = |A|^2 \int_{-\infty}^{\infty} dx\, e^{-\frac{(x-a)^2}{\Delta^2}} = |A|^2 \int_{\infty+a}^{\infty-a} dz\, e^{-\frac{z^2}{\Delta^2}}$

$z = x - a$

$$1 = |A|^2 \cdot \sqrt{\pi \cdot \Delta^2} \Rightarrow |A| = \frac{1}{(\pi\Delta^2)^{1/4}}$$

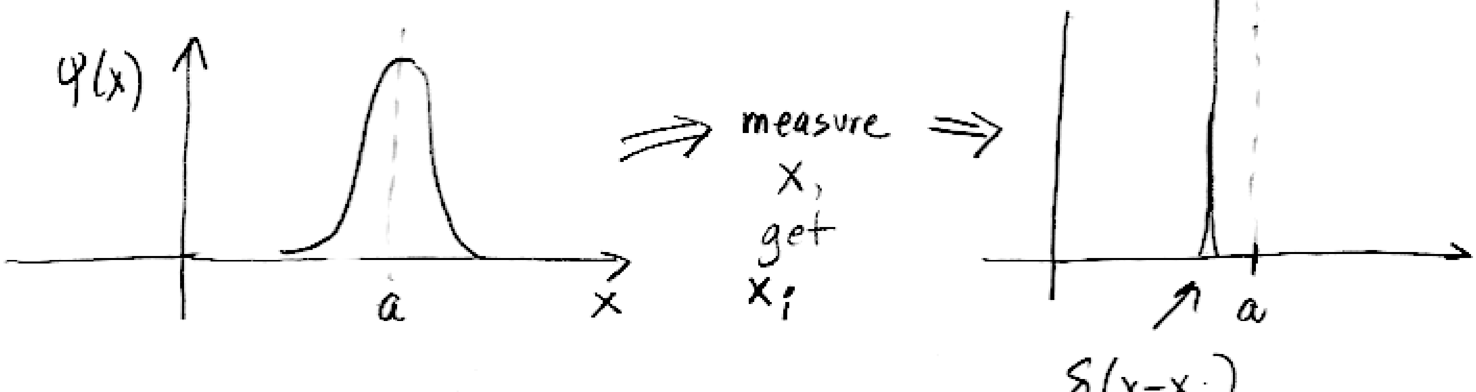
and so $$\psi(x) = \langle x|\psi\rangle = \frac{1}{(\pi\Delta^2)^{1/4}} e^{-\frac{(x-a)^2}{2\Delta^2}}$$

Probability of finding particle between x and $x+dx$ is:

$$P(x)dx = |\psi(x)|^2 dx = \frac{1}{(\pi\Delta^2)^{1/2}} e^{\frac{-(x-a)^2}{\Delta^2}} dx$$

This still peaks at $x=a$, but now the FWHM is bigger... $\text{FWHM} = \sqrt{2}\cdot 2.35 \cdot \Delta$
$= 4\sqrt{\ln 2}\,\Delta = 3.33\Delta$

Think of measurement of $\underset{\sim}{x}$:



The average, or expectation value, of x will be

$$\langle \underset{\sim}{x} \rangle = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} x_i \quad \left(\text{from } N \text{ measurements}\right)$$

$$= \langle \psi | \underset{\sim}{x} | \psi \rangle = \int_{-\infty}^{\infty} \langle \psi | x \rangle dx \langle x | \underset{\sim}{x} | \psi \rangle$$

$$\langle x | \underset{\sim}{x} | \psi \rangle = \int_{-\infty}^{\infty} \langle x | \underset{\sim}{x} | x' \rangle dx' \langle x' | \psi \rangle = \int_{-\infty}^{\infty} x\,\delta(x-x')\,dx'\,\psi(x') = x\,\psi(x)$$

$$\int_{-\infty}^{\infty} dx\, x^2 e^{-\alpha x^2} = -\frac{d}{d\alpha}\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} = \frac{-d}{d\alpha}\sqrt{\frac{\pi}{\alpha}}$$

$$= -\sqrt{\pi}\left(-\tfrac{1}{2}\right)\alpha^{-3/2} = \frac{1}{2\alpha}\left(\frac{\pi}{\alpha}\right)^{1/2}$$ (see p. 659 A.2.3)

and

$$\int_{-\infty}^{\infty} dx\, x^2 e^{\frac{-(x-a)^2}{\Delta^2}} = \int_{-\infty-a}^{\infty-a} dy\,(y^2 + \cancel{2ay}^{\,0} + a^2)\, e^{-\frac{y^2}{\Delta^2}}$$

$y = x - a$

$x = y + a$

$$= (\pi\Delta^2)^{1/2}\left(a^2 + \frac{\Delta^2}{2}\right)$$

$$\langle\psi|\underset{\sim}{x}^2|\psi\rangle = \frac{1}{(\cancel{\pi\Delta^2})^{1/2}}\cdot(\cancel{\pi\Delta^2})^{1/2}\left(a^2 + \frac{\Delta^2}{2}\right)$$

$$= a^2 + \tfrac{1}{2}\Delta^2$$

$$(\Delta x)^2 = \langle\psi|\underset{\sim}{x}^2|\psi\rangle - \langle\underset{\sim}{x}\rangle^2$$

$$= a^2 + \tfrac{1}{2}\Delta^2 - a^2 = \tfrac{1}{2}\Delta^2$$

$$\boxed{\Delta x = \frac{1}{\sqrt{2}}\Delta}$$

second meaning of Δ!
$\propto$ "uncertainty" Δx
of wave function.

What if we measure $\underset{\sim}{p}$, not $\underset{\sim}{x}$?

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must change variables.. need eigenfunctions of $\underset{\sim}{p}$, not $\underset{\sim}{x}$...

$$\underline{\langle x|\hat{p}|\psi\rangle} = \int_{-\infty}^{\infty} \underbrace{\langle x|\hat{p}|x'\rangle}_{\frac{\hbar}{i}\delta'(x-x')} dx' \underbrace{\langle x'|\psi\rangle}_{\psi(x')}$$

function that represents $\hat{p}|\psi\rangle$

$$= \frac{\hbar}{i}\int dx'\,\delta'(x-x')\,\psi(x')$$

$$= \frac{\hbar}{i}\psi'(x) = \frac{\hbar}{i}\frac{d\psi}{dx}$$

<u>eigenfunctions</u>: $\hat{p}|p\rangle = p|p\rangle$

or $\langle x|\hat{p}|p\rangle = p\underbrace{\langle x|p\rangle}_{\psi_p(x)}$

$$\frac{\hbar}{i}\frac{d\psi_p}{dx} = p\psi_p$$

so $\boxed{\psi_p(x) = B_p e^{\frac{ipx}{\hbar}}}$

normalization:

want $\langle p|p'\rangle = \delta(p-p')$

or $\int dx \langle p|x\rangle\langle x|p'\rangle = \delta(p-p') = \delta(p'-p)$

or $B_p^* B_{p'} \int_{-\infty}^{\infty} dx\, e^{\frac{i(p'-p)x}{\hbar}} = \delta(p'-p)$

1.10.26 p.63: $\frac{1}{2\pi}\int dk\, e^{i(x'-x)k} = \delta(x'-x)$

so $\quad B_p^* B_p \hbar \int_{-\infty}^{\infty} dz\, e^{i(p'-p)z} = 2\pi B_p^* B_{p'} \hbar\, \delta(p'-p)$

$z = x/\hbar$

and so $\quad 2\pi B_p^* B_{p'} \hbar\, \delta(p'-p) = \delta(p'-p)$

$$B_p^* B_{p'} = \frac{1}{2\pi\hbar}$$

most straightforward: $\quad B_p^* = B_{p'} = \frac{1}{\sqrt{2\pi\hbar}} = B_p = B_{p'}^*$

and so: $\boxed{\psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}}\, e^{\frac{ipx}{\hbar}}}$

now, make transformation from $x \to p$:

$$\langle p|\psi\rangle = \int \langle p|x\rangle\langle x|\psi\rangle\, dx$$

$$= \int \psi_p^*(x)\, \psi(x)\, dx = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{\frac{-ipx}{\hbar}}\, \psi(x)\, dx$$

now, suppose $\psi(x) = \frac{1}{(\pi\Delta^2)^{1/2}}\, e^{-\frac{(x-a)^2}{2\Delta^2}}$

$$\psi(p) = \frac{1}{(2\pi\hbar)^{1/2}}\, \frac{1}{(\pi\Delta^2)^{1/4}} \underbrace{\int_{-\infty}^{\infty} e^{\frac{-ipx}{\hbar}}\, e^{\frac{-(x-a)^2}{2\Delta^2}}\, dx}_{\text{"complete the square in the exponent"}}$$

$\to$ First, change variables: $\quad z = x - a$

$x = z + a$

$$\psi(p) = \frac{1}{(2\pi\hbar)^{1/2}} \frac{1}{(\pi\Delta^2)^{1/4}} e^{-\frac{ipa}{\hbar}} \int_{-\infty}^{\infty} e^{-\frac{ipz}{\hbar}} e^{-\frac{z^2}{2\Delta^2}} dz$$

→ Second, "complete the square" in the exponent

$$-\frac{ipz}{\hbar} - \frac{z^2}{2\Delta^2} = -\frac{1}{2\Delta^2}\left(z^2 + \frac{2\Delta^2 ip}{\hbar} z + \left(\frac{\Delta^2 ip}{\hbar}\right)^2 - \left(\frac{\Delta^2 ip}{\hbar}\right)^2\right)$$

$$= -\frac{1}{2\Delta^2}\left(z + \frac{2\Delta^2 ip}{\hbar}\right)^2 - \frac{\Delta^2 p^2}{2\hbar^2}$$

$$\psi(p) = \frac{1}{(2\pi\hbar)^{1/2}} \frac{1}{(\pi\Delta^2)^{1/4}} e^{-\frac{ipa}{\hbar} - \frac{\Delta^2 p^2}{2\hbar^2}} \underbrace{\int_{-\infty}^{\infty} dz\, e^{-\frac{1}{2\Delta^2}\left(z + \frac{2\Delta^2 ip}{\hbar}\right)^2}}_{\sqrt{2\pi\Delta^2}}$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{(\pi\Delta^2)^{1/4}} \sqrt{2\pi\Delta^2}\, e^{-\frac{ipa}{\hbar} - \frac{\Delta^2 p^2}{2\hbar^2}}$$

$$\boxed{\psi(p) = \left(\frac{\Delta^2}{\hbar^2\pi}\right)^{1/4} e^{-\frac{ipa}{\hbar} - \frac{\Delta^2 p^2}{2\hbar^2}}}$$

$$\langle p \rangle = \int_{\infty}^{-\infty} dp\, p\, |\psi(p)|^2 = 0 \quad (!)$$

$$\langle\psi|(p - \underbrace{\langle p\rangle}_{0})^2|\psi\rangle = \langle\psi|p^2|\psi\rangle = \int_{-\infty}^{\infty} dp\, p^2 |\psi(p)|^2$$

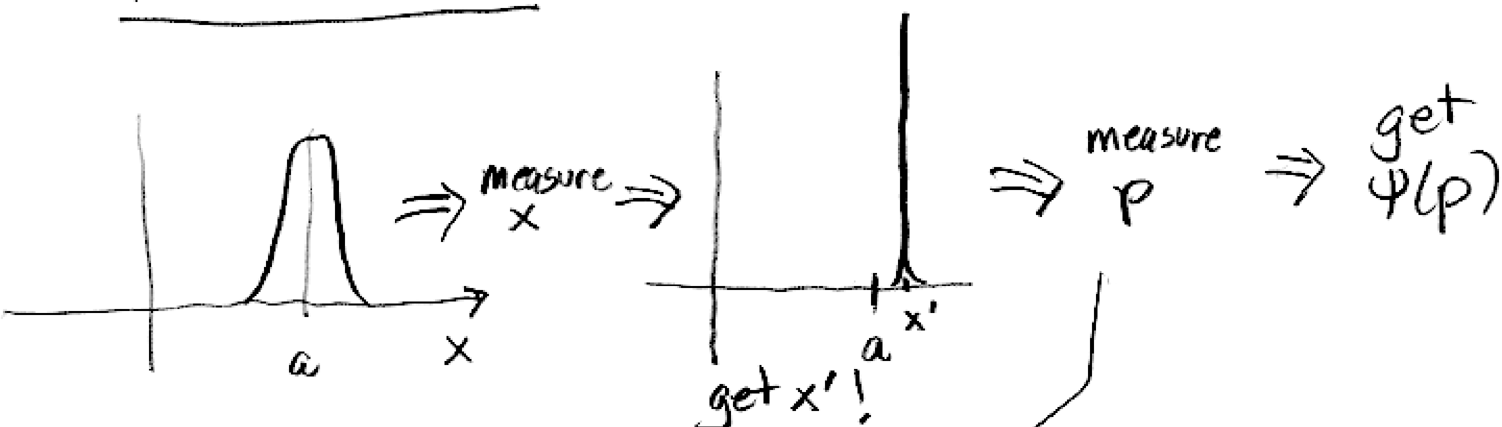
$$(\Delta p)^2 = \left(\frac{1}{\sqrt{2}} \frac{\hbar}{\Delta}\right)^2 \Rightarrow \Delta p = \frac{1}{\sqrt{2}} \frac{\hbar}{\Delta}$$

note:

$$\boxed{\Delta x \Delta p = \frac{1}{\sqrt{2}}\Delta \cdot \frac{1}{\sqrt{2}}\frac{\hbar}{\Delta} = \frac{\hbar}{2}}$$

more generally: $\Delta x \Delta p \geq \hbar/2$
(Heisenberg Uncertainty Principal)

<u>Measurements</u>:

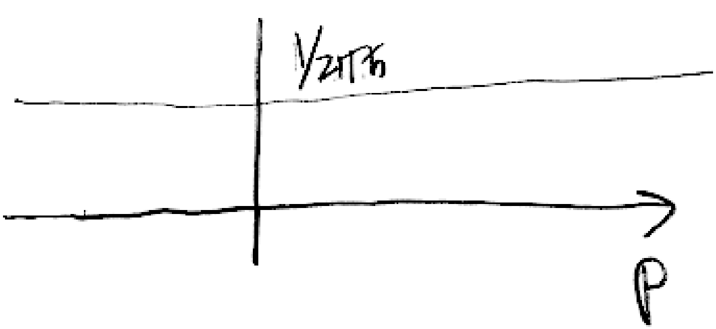


$$\psi(p) = \int \langle p|x\rangle\langle x|x'\rangle dx$$

$$= \frac{1}{(2\pi\hbar)^{1/2}}\int e^{-ipx}\,\delta(x-x')dx$$

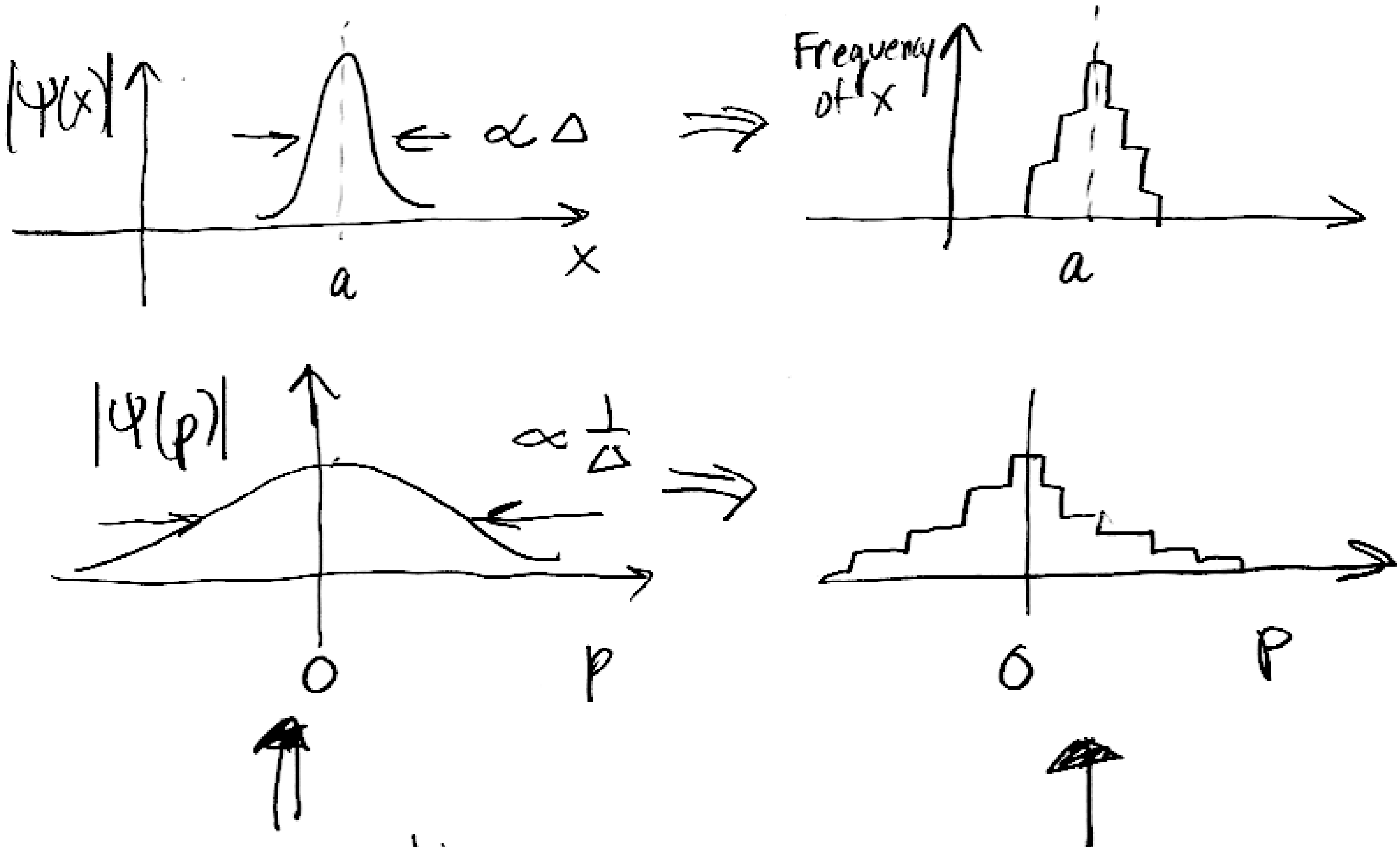
$$= \frac{1}{(2\pi\hbar)^{1/2}} e^{-ipx'}$$

$$|\psi(p)|^2 = \frac{1}{2\pi\hbar} \quad \text{(flat)}.$$



$$\boxed{\text{ALL MOMENTUM EQUALLY LIKELY}}$$

Uncertainty Principle Really Means



imagine setting up this wave function repeatedly, alternately measuring $\underset{\sim}{x}$, then $\underset{\sim}{p}$, etc.

for these frequency distributions, $\Delta x \Delta p \gtrsim \hbar/2$