Compatible & Incompatible Varables

$$|\psi\rangle \Rightarrow \underset{\substack{\text{get } \omega_i \\ \text{Prob} = |\langle \omega_i|\psi\rangle|^2 \\ \text{non-degenerate}}}{\text{Measure } \underset{\sim}{\Omega},} \Rightarrow |\omega_i\rangle \Rightarrow \underset{\substack{\text{get } \lambda_j \\ \text{(Relative) Prob} = |\langle \lambda_j|\omega_i\rangle|^2 \\ \text{non-degenerate}}}{\text{Measure } \underset{\sim}{\Lambda},} \Rightarrow |\lambda_j\rangle$$

↑ initial state

Discussion Points: Suppose we repeat measurements many times, always starting from the same (or identically prepared) initial state $|\psi\rangle$

(1) Can we always get same particular ω_i?
In general, no, will get a variety of ω_i's, each with probability $P(\omega_i) = |\langle \omega_i|\psi\rangle|^2$. Of course, when $|\psi\rangle$ itself is an eigenstate, then we'll always get the corresponding eigenvalue!

(2) Given that a <u>particular</u> ω_i results from the first measurement, when will a specific λ_j always result from the second measurement?

$\Rightarrow$ when $|\langle \lambda_j|\omega_i\rangle|^2 = 1$

or $|\omega_i\rangle$ also eigenket of $\underset{\sim}{\Lambda}$ with eigenvalue λ_j

so $\underset{\sim}{\Omega} |\omega_i \lambda_j\rangle = \omega_i |\omega_i \lambda_j\rangle$

and $\underset{\sim}{\Lambda} |\omega_i \lambda_j\rangle = \lambda_j |\omega_i \lambda_j\rangle$

further $\underset{\sim}{\Lambda}\underset{\sim}{\Omega} |\omega_i \lambda_j\rangle = \underset{\sim}{\Lambda}\omega_i |\omega_i \lambda_j\rangle = \omega_i \underset{\sim}{\Lambda} |\omega_i \lambda_j\rangle = \omega_i \lambda_j |\omega_i \lambda_j\rangle$

and $\underset{\sim}{\Omega}\underset{\sim}{\Lambda} |\omega_i \lambda_j\rangle = \underset{\sim}{\Omega}\lambda_j |\omega_i \lambda_j\rangle = \lambda_j \underset{\sim}{\Omega} |\omega_i \lambda_j\rangle = \lambda_j \omega_i |\omega_i \lambda_j\rangle$

so $(\underset{\sim}{\Omega}\underset{\sim}{\Lambda} - \underset{\sim}{\Lambda}\underset{\sim}{\Omega}) |\omega_i \lambda_j\rangle = 0$ ☆

How can we get this?

(A) Obvious: $[\underset{\sim}{\Omega}, \underset{\sim}{\Lambda}] = 0$ always

then.. $\underset{\sim}{\Omega}$ + $\underset{\sim}{\Lambda}$ are <u>compatible</u>.

Converse....

(B) $[\underset{\sim}{\Omega}, \underset{\sim}{\Lambda}] =$ something obviously non-zero

"<u>Incompatible</u>"

example: $\underset{\sim}{X}$ and $P = \hbar \underset{\sim}{k}$, $\underset{\sim}{k} \doteq \frac{1}{i}\frac{d}{dx}$

so, $[\underset{\sim}{X}, P] |f\rangle \doteq [x, \frac{\hbar}{i}\frac{d}{dx}] f(x)$

$\doteq \left(x \frac{\hbar}{i}\frac{d}{dx} - \frac{\hbar}{i}\frac{d}{dx} x\right) f(x)$

$\doteq \cancel{x \frac{\hbar}{i}\frac{df}{dx}} - \frac{\hbar}{i}\left(f(x) + \cancel{x \frac{df}{dx}}\right)$

$[\underset{\sim}{x}, P] |f\rangle \doteq -\frac{\hbar}{i} f(x)$

or $\boxed{[\underset{\sim}{X}, P] = -\frac{\hbar}{i} = i\hbar}$ famous incompatible variables

... origin of the Heisenberg uncertainty principle...

Example

$$\underset{\sim}{\Omega} \doteq \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \underset{\sim}{\Lambda} = \begin{pmatrix} a & b & d \\ b^* & c & b^* \\ d & b & a \end{pmatrix}$$

(unitary...)
aka, "symmetry operator"

$$\underset{\sim}{\Omega}\underset{\sim}{\Lambda} - \underset{\sim}{\Lambda}\underset{\sim}{\Omega} \stackrel{?}{=} 0$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} a & b & d \\ b^* & c & b^* \\ d & b & a \end{pmatrix} = \begin{pmatrix} d & b & a \\ b^* & c & b^* \\ a & b & d \end{pmatrix}$$

$$\begin{pmatrix} a & b & d \\ b^* & c & b^* \\ d & b & a \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} d & b & a \\ b^* & c & b^* \\ a & b & d \end{pmatrix}$$

but note:

$$\underset{\sim}{\Omega}\underset{\sim}{\Lambda} - \underset{\sim}{\Lambda}\underset{\sim}{\Omega} = 0$$

$$\underset{\sim}{\Lambda}^{-1}\underset{\sim}{\Omega}\underset{\sim}{\Lambda} - \underset{\sim}{\Omega} = 0 \qquad \underset{\sim}{\Lambda}^{-1} = \underset{\sim}{\Lambda}^{+} \text{ (unitary)}$$

$$\underset{\sim}{\Lambda}^{+}\underset{\sim}{\Omega}\underset{\sim}{\Lambda} = \underset{\sim}{\Omega} \longleftarrow$$

$\underset{\sim}{\Omega}$ is "invariant" under $\underset{\sim}{\Lambda}$.

$\underset{\sim}{\Lambda}$ is a "symmetry operator"

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

obvious eigenvector.

$$\begin{vmatrix} \lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & \lambda \end{vmatrix} = \lambda \begin{vmatrix} 1-\lambda & 0 \\ 0 & \lambda \end{vmatrix} + 1 \begin{vmatrix} 0 & 1-\lambda \\ 1 & 0 \end{vmatrix} = 0$$

$$\lambda(1-\lambda)\lambda - (1-\lambda) = 0$$

$$(\lambda^2-1)(1-\lambda) = 0 \qquad \lambda = \pm 1, 1$$

given $|\omega_3\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\qquad \omega_3 = 1$

$|\omega_1\rangle, |\omega_2\rangle$ of form $\begin{pmatrix} \alpha \\ 0 \\ \beta \end{pmatrix}$

± 1: $\begin{pmatrix} \mp 1 & 0 & 1 \\ 0 & 1\mp 1 & 0 \\ 1 & 0 & \mp 1 \end{pmatrix} \begin{pmatrix} \alpha \\ 0 \\ \beta \end{pmatrix}$ $\qquad +1: -\alpha + \beta = 0, \quad \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$

$\qquad -1 \quad \alpha + \beta = 0, \quad \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$

A transformation that diagonalizes $\underset{\sim}{\Omega}$ is:

$$\underbrace{\begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix}}_{\underset{\sim}{U}^{\dagger}} \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{\underset{\sim}{\Omega}} \overbrace{\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}}^{\underset{\sim}{U}}$$

$$\begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix} \left(-1 \cdot \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix} + 1 \cdot \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right)$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

now:

$|\psi\rangle \Rightarrow$ Measure $\underset{\sim}{\Omega}$ get: $\Rightarrow$ $-1 : |\omega_1\rangle$; $+1 : \dfrac{|\omega_2\rangle\langle\omega_2|\psi\rangle + |\omega_3\rangle\langle\omega_3|\psi\rangle}{\sqrt{|\langle\omega_2|\psi\rangle|^2 + |\langle\omega_3|\psi\rangle|^2}}$ $\Rightarrow$ Measure $\underset{\sim}{\Lambda}$

-1, prob $|\langle\omega_1|\psi\rangle|^2$

$$= \left| \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \right|^2$$

$+1$, prob: a) $1 - |\langle\omega_1|\psi\rangle|^2$

b) $|\langle\omega_2|\psi\rangle|^2 + |\langle\omega_3|\psi\rangle|^2$

$+1$ is ambiguous!

Transform $\underset{\sim}{\Lambda}$ into eigenbasis of $\underset{\sim}{\Omega}$:

$$\begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a & b & d \\ b^* & c & b^* \\ d & b & a \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}}(a-d) & \frac{1}{\sqrt{2}}(a+d) & b \\ 0 & \sqrt{2}\, b^* & c \\ \frac{-1}{\sqrt{2}}(a-d) & \frac{1}{\sqrt{2}}(a+d) & b \end{pmatrix}$$

$$\underset{\sim}{\Lambda} \doteq \begin{pmatrix} a-d & 0 & 0 \\ 0 & a+d & \sqrt{2}\, b \\ 0 & \sqrt{2}\, b^* & c \end{pmatrix} \leftarrow$$

measuring $\underset{\sim}{\Lambda}$ can "break" degeneracy...

- need only diagonalize 2x2 piece...
- any linear combinations of $|\omega_2\rangle$ and $|\omega_3\rangle$ are still eigenkets of $\underset{\sim}{\Omega}$ with eigenvalue $\omega_2 = \omega_3 = +1$

$$\begin{vmatrix} a+d-\lambda & \sqrt{2}\, b^* \\ \sqrt{2}\, b^* & c-\lambda \end{vmatrix} = (a+d-\lambda)(c-\lambda) - 2|b|^2 = 0$$

$$\lambda^2 - (a+c+d)\lambda + c(a+d) - 2|b|^2 = 0$$

<u>Discriminant</u>: $\overbrace{(a+c+d)}^{(a+d\,+c)}{}^2 - 4c(a+d) + 8|b|^2$

$$= (a+d)^2 + (c)^2 + 2c(a+d) - 4c(a+d) + 8|b|^2$$

$$= (a+d-c)^2 + 8|b|^2$$

so $$\lambda = \frac{(a+c+d) \pm \left[(a+d-c)^2 + 8|b|^2\right]^{1/2}}{2}$$

$$\lambda = \frac{1}{2}(a+d+c) \pm \left[\left(\frac{a+d-c}{2}\right)^2 + (\sqrt{2}|b|)^2\right]^{1/2}$$

<u>Midterm</u>: $a = \frac{2}{3}b \quad c = b \quad d = \frac{1}{3}b \quad b^* = b$

$$\lambda = \frac{1}{2}\left(\frac{2}{3} + \frac{1}{3} + 1\right)b \pm \left[\left(\cancel{\frac{\frac{2}{3}+\frac{1}{3}-1}{2}}_{\,0}\right)^2 b + (\sqrt{2}\, b)^2\right]^{1/2}$$

$$\lambda = b \pm \sqrt{2}\, b$$

<u>Eigenvectors</u>: $\lambda = b + \sqrt{2}b$: $\begin{pmatrix} \frac{1}{3}b - b - \sqrt{2}b & 0 & 0 \\ 0 & -\sqrt{2}b & \sqrt{2}b \\ 0 & \sqrt{2}b & -\sqrt{2}b \end{pmatrix} \begin{pmatrix} 0 \\ \alpha \\ \beta \end{pmatrix} = 0$

$\alpha = \beta$, $\begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ representation in $\underset{\sim}{\Omega}$ eigenbasis

meaning $\frac{1}{\sqrt{2}}\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix}$ in original basis

$\lambda = b + \sqrt{2}b$

$\lambda = b - \sqrt{2}b$: $\begin{pmatrix} \frac{1}{3}b - b + \sqrt{2}b & 0 & 0 \\ 0 & \sqrt{2}b & \sqrt{2}b \\ 0 & \sqrt{2}b & \sqrt{2}b \end{pmatrix}\begin{pmatrix} 0 \\ \alpha \\ \beta \end{pmatrix} = 0$

$\alpha = -\beta$, $\begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$ representation in $\underset{\sim}{\Omega}$ eigenbasis.

meaning $\frac{1}{\sqrt{2}}\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{pmatrix}$ in original basis

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(Ideal)

$|\psi\rangle \rightarrow$ measure $\underset{\sim}{\Omega}$: $\omega_1 = -1 \rightarrow |\omega_1\rangle \rightarrow$ measure $\underset{\sim}{\Lambda}$: $\frac{1}{3}b \rightarrow$ still get $\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$; $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

measure $\underset{\sim}{\Omega}$: $\omega_2 = \omega_3 = +1 \rightarrow \dfrac{|\omega_2\rangle\langle\omega_2|\psi\rangle + |\omega_3\rangle\langle\omega_3|\psi\rangle}{\sqrt{|\langle\omega_2|\psi\rangle|^2 + |\langle\omega_3|\psi\rangle|^2}} \rightarrow$ measure $\underset{\sim}{\Lambda}$:

$b+\sqrt{2}b \rightarrow$ get $\begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix}$; $\begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

$b-\sqrt{2}b \rightarrow$ get $\begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{pmatrix}$; $\begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

original basis

first eigenbasis of $\underset{\sim}{\Omega}$

still eigenkets of $\underset{\sim}{\Omega}$, eigenvalue $+1$

Since there are 3 distinct combinations of eigenvalues:

$(-1, \frac{1}{3}b)$

$(+1, b+\sqrt{2}b)$

$(+1, b-\sqrt{2}b)$

$\underset{\sim}{\Omega}, \underset{\sim}{\Lambda}$ form a "Complete Set of Commuting Observables"