Many Matrices Can Represent the Same Linear Op

$$\underset{\sim}{\Omega} \doteq \begin{pmatrix} \Omega_{11} & \Omega_{12} & \cdots \\ \Omega_{21} & \Omega_{22} & \\ \vdots & & \ddots \end{pmatrix}, \quad \Omega_{ij} = \langle i|\underset{\sim}{\Omega}|j\rangle, \quad \underset{\sim}{\Omega} = \sum_{ij} |i\rangle \Omega_{ij} \langle j|$$

one basis

$$\underset{\sim}{or} \quad \underset{\sim}{\Omega} \doteq \begin{pmatrix} \Omega'_{11} & \Omega'_{12} & \cdots \\ \Omega'_{21} & \Omega'_{22} & \\ \vdots & & \end{pmatrix}, \quad \Omega'_{ij} = \langle i'|\underset{\sim}{\Omega}|j'\rangle, \quad \underset{\sim}{\Omega} = \sum_{ij} |i'\rangle \Omega'_{ij} \langle j'|$$

and $\Omega_{ij} = \langle i|\underset{\sim}{\Omega}|j\rangle = \sum_{k\ell} \underbrace{\langle i|k'\rangle}_{U^{\dagger}_{ik}} \underbrace{\langle k'|\Omega|\ell'\rangle}_{\Omega'_{k\ell}} \underbrace{\langle \ell'|j\rangle}_{U_{\ell j}}$

$$\Omega_{ij} = \sum_{k\ell} U^{\dagger}_{ik} \Omega'_{k\ell} U_{\ell j}$$

$\boxed{\text{Different (matrix) representations of the same } \underset{\sim}{\Omega} \text{ are related by a unitary transformation.}}$

"Fundamental" properties of $\underset{\sim}{\Omega}$: • Trace; • Determinant

Determinant:

0 when ... columns linearly dependent

example: $\begin{pmatrix} a & a \\ b & b \end{pmatrix}$; $\begin{vmatrix} a & a \\ b & b \end{vmatrix} = ab - ab = 0$

note also: $\begin{pmatrix} a & a \\ b & b \end{pmatrix} \left[c \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] = 0$ (no inverse!)

→ when det(Matrix) = 0,

1) No inverse, meaning non-trivial vectors are mapped into 0

2) "Volume" of matrix is 0

Eigenvalues (these are fundamental too)

There will (usually) be one matrix representation of $\underset{\sim}{\Omega}$ that is diagonal, namely ...

$$\underset{\sim}{\Omega} \doteq \begin{pmatrix} \omega_1 & 0 & 0 & \cdots \\ 0 & \omega_2 & 0 & \\ 0 & 0 & \omega_3 & \\ \vdots & \vdots & & \ddots & \\ & & & & \omega_n \end{pmatrix}$$

the basis that diagonalizes $\underset{\sim}{\Omega}$ is the "eigenbasis," $\{|\omega_1\rangle, |\omega_2\rangle, \ldots |\omega_n\rangle\}$ and $\langle\omega_i|\underset{\sim}{\Omega}|\omega_j\rangle = \omega_i\delta_{ij}$

$\langle\omega_i|\omega_j\rangle = \delta_{ij}$

note

$$\begin{pmatrix} \omega_1 & 0 & & 0 \\ 0 & \omega_2 & & \\ \vdots & & \ddots & \\ & & & \omega_n \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} = \omega_1\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \qquad \begin{pmatrix} \omega_1 & 0 & & \\ 0 & \omega_2 & & \\ \vdots & & \ddots & \\ & & & \omega_n \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} = \omega_2\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$$

operator × vector = number × $\binom{\text{same}}{\text{vector}}$; $\underset{\sim}{\Omega}|\omega_i\rangle = \omega_i|\omega\rangle$

These are the eigenvectors, represented in the eigenbasis. Note that taking R.H.S. to L.H.S. ...

$$\begin{pmatrix} \omega_1-\omega_1=0 & 0 & & \\ 0 & \omega_2 & & \\ & & \ddots & \\ & & & \omega_n \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \omega_2 & & \\ \vdots & & \ddots & \\ 0 & & & \omega_n \end{pmatrix}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} = 0$$

- no inverse
- det $= 0 \times \omega_2 \times \omega_3 \times \cdots \omega_n = 0$

It would be great if we all only had to ever work with diagonal representations of matrices. But often (particularly in QM), the set up of a problem hands you a representation of a matrix that is not diagonal. How to diagonalize $\underset{\sim}{\Omega}$?

Meaning: $\omega_i\delta_{ij} = \langle\omega_i|\underset{\sim}{\Omega}|\omega_j\rangle = \sum_{k\ell}\langle\omega_i|k\rangle\langle k|\underset{\sim}{\Omega}|\ell\rangle\langle\ell|\omega_j\rangle$

$$w_i \delta_{ij} = \sum_{k\ell} \underset{U^+_{ik}}{\langle w_i | k \rangle} \Omega_{k\ell} \underbrace{\langle \ell | w_j \rangle}_{U_{\ell j}}$$

$U_{\ell j}$ ⟷ special, takes you to the eigenbasis.

Sometimes you only get $\Omega_{k\ell}$, which isn't generally digonal! But you'd like to get the eigenvalues anyway. How?

Start from: $\Omega |w_i\rangle = w_i |w_i\rangle$

$\sum_k |k\rangle\langle k|$

$\sum_\ell |\ell\rangle\langle \ell|$ ← "random basis"

$$\sum_{k\ell} |k\rangle\langle k| \Omega |\ell\rangle\langle \ell | w_i\rangle = \sum_k w_i |k\rangle\langle k | w_i\rangle$$

"eigen-vector"

or
$$\begin{pmatrix} \Omega_{11} & \Omega_{12} & \cdots & \Omega_{1n} \\ \Omega_{21} & \Omega_{22} & & \\ \vdots & & \ddots & \\ \Omega_{n1} & & & \Omega_{nn} \end{pmatrix} \begin{pmatrix} \langle 1|w_i\rangle \\ \langle 2|w_i\rangle \\ \vdots \\ \langle n|w_i\rangle \end{pmatrix} = w_i \begin{pmatrix} \langle 1|w_i\rangle \\ \langle 2|w_i\rangle \\ \vdots \\ \langle n|w_i\rangle \end{pmatrix}$$

not all zeros

take to left

$$\underbrace{\begin{pmatrix} \Omega_{11}-w_i & \Omega_{12} & \cdots & \Omega_{1n} \\ \Omega_{21} & \Omega_{22}-w_i & & \\ \vdots & & \ddots & \\ \Omega_{n1} & & & \Omega_{nn}-w_i \end{pmatrix}}_{} \underbrace{\begin{pmatrix} \langle 1|w_i\rangle \\ \langle 2|w_i\rangle \\ \vdots \\ \langle n|w_i\rangle \end{pmatrix}}_{} = 0$$

- must have no inverse
- det is ZERO.

non-trivial (not all zeros)

$$\det \begin{pmatrix} \Omega_{11}-\omega & \Omega_{12} & \cdots & \Omega_{1n} \\ \Omega_{21} & \Omega_{22}-\omega & & \\ \vdots & & \ddots & \\ \Omega_{n1} & & & \Omega_{nn}-\omega \end{pmatrix} = 0$$

$\leftarrow$ dropped subscript from ω

will be an n^{th} order polynomial, known as the "<u>characteristic</u> <u>equation</u>"

$$P^n(\omega) = \sum_{m=0}^{n} c_m \omega^m = 0$$

$\leftarrow$ generally, n roots $\{\omega_1, \omega_2, \omega_3, \ldots, \omega_n\}$

one solves the characteristic equation for the eigenvalues (n of them) first. Once you've got the n eigenvalues, $\{\omega_1, \omega_2, \omega_3, \ldots, \omega_n\}$, you then plug back in to the equation:

$$\begin{pmatrix} \Omega_{11}-\omega_i & \Omega_{12} & \cdots & \Omega_{1n} \\ \Omega_{21} & \Omega_{22}-\omega_i & & \\ \vdots & & \ddots & \\ \Omega_{n1} & & & \Omega_{nn}-\omega_i \end{pmatrix} \begin{pmatrix} \langle 1|\omega_i\rangle \\ \langle 2|\omega_i\rangle \\ \vdots \\ \langle n|\omega_i\rangle \end{pmatrix} = 0$$

<u>suppose</u> $\underset{\sim}{\Omega} \doteq \begin{pmatrix} 1 & \sqrt{2}i \\ -\sqrt{2}i & 2 \end{pmatrix}$ in some "random" basis

eigenvalues: $\begin{vmatrix} 1-\omega & \sqrt{2}i \\ -\sqrt{2}i & 2-\omega \end{vmatrix} = (1-\omega)(2-\omega) - 2i(-i)$

$= \cancel{2} - 3\omega + \omega^2 - \cancel{2} = 0$

$= \underbrace{\omega^2 - 3\omega}_{P^2(\omega)} = 0 = \omega(\omega-3)$

$\boxed{\omega_1 = 0, \ \omega_2 = 3}$

<u>note</u>: $\operatorname{Tr}(\underset{\sim}{\Omega}) = 1 + 2 = 3$

<u>in eigenbasis</u>

$$\underset{\sim}{\Omega} = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}, \quad \operatorname{Tr}(\underset{\sim}{\Omega}) = \omega_1 + \omega_2 = 3$$

$$\det(\underset{\sim}{\Omega}) = \begin{vmatrix} 1 & \sqrt{2}i \\ -\sqrt{2}i & 2 \end{vmatrix} = 2 - (\sqrt{2})(-\sqrt{2})i^2 = 2 - 2 = 0$$

$$= \begin{vmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{vmatrix} = \omega_1 \cdot \omega_2 = 0 \cdot 3 = 0.$$

Now: what are the eigenvectors?

$$\omega_1: \begin{pmatrix} 1-\omega_1 & \sqrt{2}i \\ -\sqrt{2}i & 2-\omega_1 \end{pmatrix} \begin{pmatrix} \langle 1|\omega_1\rangle \\ \langle 2|\omega_1\rangle \end{pmatrix} = 0$$

call: $\langle 1|\omega_1\rangle = U_{11}$, $\langle 2|\omega_1\rangle = U_{21}$

$$\begin{pmatrix} 1 & \sqrt{2}i \\ -\sqrt{2}i & 2 \end{pmatrix} \begin{pmatrix} U_{11} \\ U_{21} \end{pmatrix} = 0, \quad \begin{aligned} &U_{11} + \sqrt{2}i\,U_{21} = 0 \quad \frac{U_{21}}{U_{11}} = \frac{-1}{\sqrt{2}i} \\ &-\sqrt{2}i\,U_{11} + 2U_{21} = 0 \quad \frac{U_{21}}{U_{11}} = \frac{-\sqrt{2}i}{-2} \end{aligned}$$

both!

$$\boxed{\frac{U_{21}}{U_{11}} = \frac{i}{\sqrt{2}}}$$

want this vector to be <u>unit</u> <u>length</u>

$$(U_{11}^* \; U_{21}^*) \begin{pmatrix} U_{11} \\ U_{21} \end{pmatrix} = |U_{11}|^2 + |U_{21}|^2 = 1$$

$$= |U_{11}|^2 \left(1 + \left|\frac{U_{21}}{U_{11}}\right|^2\right) = 1$$

$$= |U_{11}|^2 \left(1 + \frac{1}{2}\right) = 1$$

$$= |U_{11}|^2 \times \frac{3}{2} = 1, \quad |U_{11}| = \sqrt{\frac{2}{3}}$$

with δ_1 = arbitrary real #, $\boxed{U_{11} = \sqrt{\frac{2}{3}}\, e^{i\delta_1}, \quad U_{21} = \sqrt{\frac{1}{3}}\, i e^{i\delta_1}}$

now: plug in $\omega_2 = 3$

$$\begin{pmatrix} 1-\omega_2 & \sqrt{2}i \\ -\sqrt{2}i & 2-\omega_2 \end{pmatrix} \begin{pmatrix} \langle 1|\omega_2\rangle \\ \langle 2|\omega_2\rangle \end{pmatrix} = 0$$

call $\langle 1|\omega_2\rangle = U_{12}$
$\langle 2|\omega_2\rangle = U_{22}$

$$\begin{pmatrix} -2 & \sqrt{2}i \\ -\sqrt{2}i & -1 \end{pmatrix} \begin{pmatrix} U_{12} \\ U_{22} \end{pmatrix} = 0 \Rightarrow \begin{matrix} -2U_{12} + \sqrt{2}i\,U_{22} = 0 \\ -\sqrt{2}i\,U_{12} - U_{22} = 0 \end{matrix} \quad \begin{matrix} \frac{U_{12}}{U_{22}} = \frac{-\sqrt{2}i}{-2} \\ \frac{U_{12}}{U_{22}} = \frac{1}{-\sqrt{2}i} \end{matrix}$$

both!

$$\frac{U_{12}}{U_{22}} = \frac{i}{\sqrt{2}}$$

$$\begin{pmatrix} U_{12}^* & U_{22}^* \end{pmatrix} \begin{pmatrix} U_{12} \\ U_{22} \end{pmatrix} = |U_{12}|^2 + |U_{22}|^2 = 1$$

$$|U_{22}|^2 \left(1 + \frac{|U_{12}|^2}{|U_{22}|^2}\right) = 1$$

$$|U_{22}|^2 \left(1 + \frac{1}{2}\right) = 1 = |U_{22}|^2 \times \frac{3}{2} = 1$$

$$|U_{22}|^2 = \frac{2}{3} \qquad |U_{22}| = \sqrt{\frac{2}{3}}$$

with δ_2 = arbitrary real #, $\boxed{U_{22} = \sqrt{\frac{2}{3}}\,e^{i\delta_2} \quad U_{12} = \sqrt{\frac{1}{3}}\,i\,e^{i\delta_2}}$

<u>note</u>:

$$\langle \omega_2|\omega_1\rangle = \sum_i \langle \omega_2|i\rangle\langle i|\omega_1\rangle = \begin{pmatrix} U_{12}^* & U_{22}^* \end{pmatrix} \begin{pmatrix} U_{11} \\ U_{21} \end{pmatrix}$$

$$= \begin{pmatrix} -\sqrt{\frac{1}{3}}\,i\,e^{-i\delta_2} & \sqrt{\frac{2}{3}}\,e^{-i\delta_2} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{2}{3}}\,e^{i\delta_1} \\ \sqrt{\frac{1}{3}}\,i\,e^{i\delta_1} \end{pmatrix}$$

$$= i e^{i(\delta_1 - \delta_2)} \left(-\frac{\sqrt{2}}{3} + \frac{\sqrt{2}}{3} \right) = 0$$

basis is orthonormal

That means ... the matrix formed by colums of eigenvectors is unitary:

$$\underset{\sim}{U} \doteq \begin{pmatrix} \langle 1|w_1\rangle & \langle 1|w_2\rangle \\ \langle 2|w_1\rangle & \langle 2|w_2\rangle \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} = \begin{pmatrix} \sqrt{\tfrac{2}{3}}\, e^{i\delta_1} & \sqrt{\tfrac{1}{3}}\, i e^{i\delta_2} \\ \sqrt{\tfrac{1}{3}}\, i e^{i\delta_1} & \sqrt{\tfrac{2}{3}}\, e^{i\delta_2} \end{pmatrix}$$

$$\underset{\sim}{U}^{+} \underset{\sim}{U} = \begin{pmatrix} \sqrt{\tfrac{2}{3}}\, e^{-i\delta_1} & -\sqrt{\tfrac{1}{3}}\, i e^{-i\delta_1} \\ -\sqrt{\tfrac{1}{3}}\, i e^{-i\delta_2} & \sqrt{\tfrac{2}{3}}\, e^{-i\delta_2} \end{pmatrix} \begin{pmatrix} \sqrt{\tfrac{2}{3}}\, e^{i\delta_1} & \sqrt{\tfrac{1}{3}}\, i e^{i\delta_2} \\ \sqrt{\tfrac{1}{3}}\, i e^{i\delta_1} & \sqrt{\tfrac{2}{3}}\, e^{i\delta_2} \end{pmatrix} = \begin{pmatrix} \tfrac{2}{3}+\tfrac{1}{3} & \overbrace{i e^{i(\delta_2-\delta_1)} \tfrac{\sqrt{2}}{3}(1-1)}^{0} \\ \underbrace{i e^{i(\delta_1-\delta_2)} \tfrac{\sqrt{2}}{3}(-1+1)}_{0} & \tfrac{1}{3}+\tfrac{2}{3} \end{pmatrix}$$

.. usually, choose δ_1, δ_2 for convenience.

<u>Th'm 9</u>: The eigenvalues of a Hermitian Operator are real.

H.O.: $\underset{\sim}{\Omega} = \underset{\sim}{\Omega}^{+}$

<u>eigenvector</u>: $\underset{\sim}{\Omega}|w\rangle = w|w\rangle$ and $\langle w|w\rangle \neq 0$ (non-trivial).

$$\langle w|\underset{\sim}{\Omega}|w\rangle = w\langle w|w\rangle$$

Adjoint that $\langle w|\underset{\sim}{\Omega}^{+} = w^{*}\langle w|$

self-adjoint

$$\langle w|\underset{\sim}{\Omega} = w^{*}\langle w|$$

$$\langle w|\underset{\sim}{\Omega}|w\rangle = w^{*}\cancel{\langle w|w\rangle} = w\,\cancel{\langle w|w\rangle}$$

$$\boxed{w^{*} = w} \quad w \text{ is real.}$$

Th'm 10 Basis of eigenkets of an H.O. is ortho-normal.

orthogonal that's the key point

normalize, by choice.

$\boxed{\text{Case A}}$ All eigenvalues are distinct; no two eigenvalues are equal (non-degenerate)

$$\underset{\sim}{\Omega}|\omega_i\rangle = \omega_i|\omega_i\rangle \qquad \underbrace{\langle\omega_j|\underset{\sim}{\Omega}^\dagger}_{\text{Hermitian}} = \omega_j^*\langle\omega_j|$$

$$\langle\omega_j|\underset{\sim}{\Omega} = \omega_j^*\langle\omega_j| = \omega_j\langle\omega_j|$$

now look at: $\langle\omega_j|\underset{\sim}{\Omega}|\omega_i\rangle$

$$= \omega_i\langle\omega_j|\omega_i\rangle$$
$$= \omega_j\langle\omega_j|\omega_i\rangle$$

not equal — must be zero!

by choice $\langle\omega_i|\omega_i\rangle = 1$

so $\langle\omega_i|\omega_j\rangle = \delta_{ij}$ (orthonormal)

$\boxed{\text{Case B}}$ some equal eigenvalues ("degenerate")

- still, $\langle\omega_i|\omega_j\rangle = 0$ when $\omega_i \neq \omega_j$

- complication comes when $\omega_i = \omega_j$ easier than you might think.

think of ....

$$\begin{pmatrix} w_1 & 0 & & & \\ 0 & w_2 & & & \\ & & \ddots & & \\ & & & \boxed{\begin{matrix} w_n & 0 & 0 \\ 0 & w_n & 0 \\ 0 & 0 & w_n \end{matrix}} & \\ & & & & \ddots \end{pmatrix}$$

- this <u>block</u> is

$$= w_n \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- identity matrix looks the same in <u>any</u> basis.

- must choose a basis (in the degenerate subspace) that is orthonormal, by Gram-Schmidt, for example.