in that case $\underset{\sim}{R}(\frac{1}{2}\pi\hat{i})$'s representation might be very complicated!

Adjoint Again

$\underset{\sim}{\Omega}^{\dagger}$ is.... best thought of as acting on the matrix representation:

1. Take transpose (flip matrix across diagonal).
2. Take complex conjugate

if $\underset{\sim}{\Omega} \doteq \begin{pmatrix} \Omega_{11} & \Omega_{12} & \cdots & \Omega_{1n} \\ \Omega_{21} & \Omega_{22} & & \\ \vdots & & \ddots & \\ \Omega_{n1} & & & \Omega_{nn} \end{pmatrix}$

$$\underset{\sim}{\Omega}^{\dagger} \doteq \begin{pmatrix} \Omega_{11}^* & \Omega_{12}^* & & \Omega_{n1}^* \\ \Omega_{12}^* & \Omega_{22}^* & & \\ & & \ddots & \\ \Omega_{1n}^* & & & \Omega_{nn}^* \end{pmatrix}$$

Rotation Matrices (and operators) have a very important property..... first...

Linear Operation = Matrix Multiplication

$\underset{\sim}{\Omega}|V\rangle$ same as: $\begin{pmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \cdots & \Omega_{1n} \\ \Omega_{21} & \Omega_{22} & & & \\ \Omega_{31} & & \ddots & & \\ \vdots & & & \ddots & \\ \Omega_{n1} & & & & \Omega_{nn} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_n \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} \Omega_{1i} V_i \\ \sum_{i=1}^{n} \Omega_{2i} V_i \\ \vdots \\ \sum_{i=1}^{n} \Omega_{ni} V_i \end{pmatrix}$

Sequential Operation = Matrix Multiplication

$$\underset{\sim}{\Lambda}\,\underset{\sim}{\Omega}|V\rangle \doteq \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \cdots & \Lambda_{1n} \\ \Lambda_{21} & \Lambda_{22} & & \vdots \\ \vdots & & & \\ \Lambda_{n1} & \cdots & & \Lambda_{nn} \end{pmatrix} \begin{pmatrix} \Omega_{11} & \Omega_{12} & \cdots & \Omega_{1n} \\ \Omega_{21} & \Omega_{22} & & \vdots \\ \vdots & & & \\ \Omega_{n1} & \cdots & & \Omega_{nn} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{pmatrix}$$

$$\boxed{\text{For Rotation Matrices, Their Adjoint is Their INVERSE}}$$

Check: first representation:

$$\underset{\sim}{R}\left(\tfrac{1}{2}\pi\hat{i}\right) \doteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \underset{\sim}{R}^{\dagger}\left(\tfrac{1}{2}\pi\hat{i}\right) \doteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\underset{\sim}{R}\left(\tfrac{1}{2}\pi\hat{i}\right)\underset{\sim}{R}^{\dagger}\left(\tfrac{1}{2}\pi\hat{i}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underset{\sim}{R}^{\dagger}\left(\tfrac{1}{2}\pi\hat{i}\right)\underset{\sim}{R}\left(\tfrac{1}{2}\pi\hat{i}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Unit Matrix

operators with the property

$$\underset{\sim}{\Omega}\,\underset{\sim}{\Omega}^{\dagger} = \underset{\sim}{\Omega}^{\dagger}\,\underset{\sim}{\Omega} = \underset{\sim}{\mathbb{1}}$$ are

called $\boxed{\text{UNITARY}}$

Projection Operators

Identity is ...
$$\begin{pmatrix} 1 & 0 & 0 & \cdots & \\ 0 & 1 & 0 & & \\ 0 & 0 & 1 & \cdots & \\ \vdots & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

Operators like
$$P_1 \doteq \begin{pmatrix} 1 & 0 & 0 & \cdots & \\ 0 & 0 & 0 & & \\ 0 & 0 & 0 & \ddots & \\ \vdots & \vdots & & & \\ & & & & 0 \end{pmatrix}$$

are very special. Note, they don't have an <u>inverse</u> in general. For the case above

$$\begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ \vdots & & & \ddots \end{pmatrix} \begin{pmatrix} 0 \\ a \\ b \\ c \\ d \\ \vdots \end{pmatrix} = 0, \text{ so, no inverse.}$$

note, however, $P_1^2 = \begin{pmatrix} 1 & 0 & \cdots \\ 0 & 0 & \\ \vdots & & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots \\ 0 & & \\ \vdots & & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots \\ 0 & 0 & \\ \vdots & & \ddots \end{pmatrix} = P_1$

$$\boxed{P_1^2 = P_1} \leftarrow$$ key relationship; if satisfied, P_1 is a projection operator

"Acting again does no good, you get it all on the first operation!"

Recipe for a projection operator:

1) find a unit vector: $|w\rangle \doteq$ say, $\begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}$ $\langle w|w\rangle = 1$

2) multiply $|w\rangle$ times its adjoint $|w\rangle\langle w|$

$$\rho_w \equiv |w\rangle\langle w| \doteq \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix} \times \begin{pmatrix} 1^*/2 & \sqrt{3}^*/2 \end{pmatrix}$$
$$= \begin{pmatrix} 1/4 & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & 3/4 \end{pmatrix}$$

3) check two ways: $\rho_w^2 = |w\rangle\underbrace{\langle w|w\rangle}_{1}\langle w| = |w\rangle\langle w|$

$$\rho_w^2 = \rho_w$$

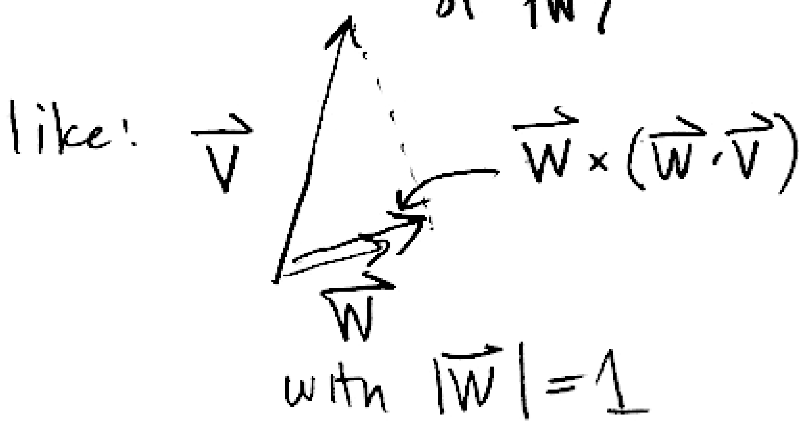
or $\begin{pmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{pmatrix}\begin{pmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{16}+\frac{3}{16}=\frac{1}{4} & \frac{\sqrt{3}}{16}+\frac{\sqrt{3}\cdot 3}{16}=\frac{1}{4}\sqrt{3} \\ \frac{\sqrt{3}}{16}+\frac{3\sqrt{3}}{16}=\frac{1}{4}\cdot\sqrt{3} & \frac{3}{16}+\frac{9}{16}=\frac{12}{16}=\frac{3}{4} \end{pmatrix}$

$$= \begin{pmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{pmatrix}$$

Projection Operators describe a subspace, and when they operate on a ket $|v\rangle$, they yield the component of $|v\rangle$ that lives in that subspace:

for example $\rho_w|v\rangle = \underbrace{|w\rangle}_{\text{in the direction of } |w\rangle}\langle w|v\rangle$

"dot" product



"Dimension" of a Projection Operator

2 unit kets: $\langle 1|1\rangle = \langle 2|2\rangle = 1$

orthogonal $\langle 1|2\rangle = \langle 2|1\rangle = 0$

$$P_{12} \equiv |1\rangle\langle 1| + |2\rangle\langle 2|$$

Is it a projection operator?

$$P_{12}^2 = \left(|1\rangle\langle 1| + |2\rangle\langle 2|\right)\left(|1\rangle\langle 1| + |2\rangle\langle 2|\right)$$

$$= |1\rangle\underbrace{\langle 1|1\rangle}_{1}\langle 1| + |1\rangle\underbrace{\langle 1|2\rangle}_{0}\langle 2| + |2\rangle\underbrace{\langle 2|1\rangle}_{0}\langle 1| + |2\rangle\underbrace{\langle 2|2\rangle}_{1}\langle 2|$$

$$= |1\rangle\langle 1| + |2\rangle\langle 2| \quad \checkmark \text{ yes}$$

---

when: $|1\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad |2\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$|1\rangle\langle 1| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1^* & 0^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad |2\rangle\langle 2| = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1^* \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$|1\rangle\langle 1| + |2\rangle\langle 2| \doteq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underset{\sim}{\mathbb{1}}$$ (since only two dimensions)

1) $Tr(P)$ = dimension, of relevant subspace.

2) For a basis of n kets, $|1\rangle, |2\rangle, \ldots, |n\rangle$ when $\sum_{i=1}^{n} |i\rangle\langle i| = \underset{\sim}{\mathbb{1}}$ we say the basis is complete, "completeness"

3) Operator: $\underset{\sim}{\Omega}$

$$\underset{\sim}{\Omega} \doteq \begin{pmatrix} \Omega_{11} & \Omega_{12} & \cdots & \Omega_{1n} \\ \Omega_{21} & \Omega_{22} & & \\ \vdots & & \ddots & \\ \Omega_{n1} & & & \Omega_{nn} \end{pmatrix}$$

in some basis $|i\rangle$

$$\underset{\sim}{\Omega} = \sum_{ij} |i\rangle \Omega_{ij} \langle j|$$

then $\langle k|\underset{\sim}{\Omega}|l\rangle = \sum_{ij} \underbrace{\langle k|i\rangle}_{\delta_{ki}} \Omega_{ij} \underbrace{\langle j|l\rangle}_{\delta_{jl}}$

pick k, l

$$\boxed{\langle k|\underset{\sim}{\Omega}|l\rangle = \Omega_{kl}}$$

aka $\Omega_{ij} = \langle i|\underset{\sim}{\Omega}|j\rangle$

4) Matrix Multiplication:

$$(\underset{\sim}{\Omega}\underset{\sim}{\Lambda})_{ij} = \langle i|\underset{\sim}{\Omega}\underset{\sim}{\Lambda}|j\rangle$$

$$\langle i|\underset{\sim}{\Omega}\,\underset{\sim}{\mathbb{1}}\,\underset{\sim}{\Lambda}|j\rangle$$

$$\underset{\sim}{\mathbb{1}} = \sum_{k=1}^{n} |k\rangle\langle k|$$

$$= \langle i|\underset{\sim}{\Omega} \sum_{k=1}^{n} |k\rangle\langle k| \underset{\sim}{\Lambda}|j\rangle$$

$$= \sum_{k=1}^{n} \langle i|\underset{\sim}{\Omega}|k\rangle\langle k|\underset{\sim}{\Lambda}|j\rangle = \sum_{k} \Omega_{ik}\Lambda_{kj}$$

Adjoints

$$\underset{\sim}{\Omega} \doteq \begin{pmatrix} \Omega_{11} & \Omega_{12} & \cdots & \Omega_{1n} \\ \Omega_{21} & \Omega_{22} & & \\ \vdots & \vdots & \ddots & \\ \Omega_{n1} & \cdots & & \Omega_{nn} \end{pmatrix} \qquad \underset{\sim}{\Omega}^{\dagger} \doteq \begin{pmatrix} \Omega_{11}^{*} & \Omega_{21}^{*} & \cdots & \Omega_{n1}^{*} \\ \Omega_{12}^{*} & \Omega_{22}^{*} & & \\ & & \ddots & \\ \Omega_{1n}^{*} & & & \Omega_{nn}^{*} \end{pmatrix}$$

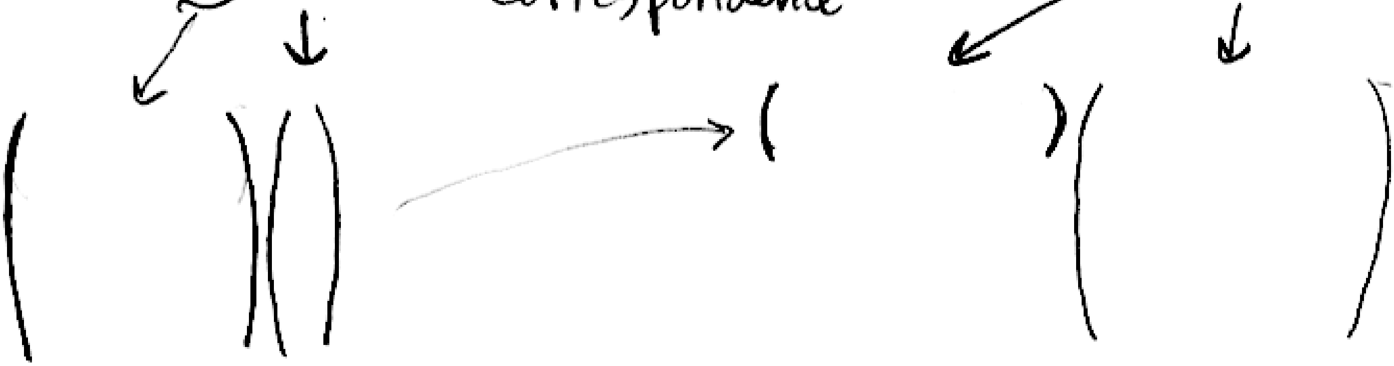
$$\Omega^{\dagger}_{ij} = \Omega^{*}_{ji}$$

other ways:

$$\underset{\sim}{\Omega} = \sum_{ij} |i\rangle \Omega_{ij} \langle j| \qquad \langle k|\underset{\sim}{\Omega}|\ell\rangle = \Omega_{k\ell}$$

$$\underset{\sim}{\Omega}^{\dagger} = \sum_{ij} |j\rangle \Omega^{*}_{ij} \langle i| \qquad \langle k|\underset{\sim}{\Omega}^{\dagger}|\ell\rangle = \Omega^{*}_{\ell k}$$

ket: $\underset{\sim}{\Omega}|v\rangle \xrightarrow[\text{correspondence}]{\text{dual}}$ bra $\langle v|\underset{\sim}{\Omega}^{\dagger}$



$$\underset{\sim}{\Lambda}\,\underset{\sim}{\Omega}|v\rangle \longrightarrow \langle v|\underset{\sim}{\Omega}^{\dagger}\underset{\sim}{\Lambda}^{\dagger}$$

so $\quad (\underset{\sim}{\Lambda}\,\underset{\sim}{\Omega})^{\dagger} = \underset{\sim}{\Omega}^{\dagger}\underset{\sim}{\Lambda}^{\dagger}$

Hermitian Operators

Comparison Table:

Complex Numbers α, β	Operators $\underset{\sim}{\Omega}, \underset{\sim}{\Lambda}$
• commute: $\alpha\beta = \beta\alpha$	• don't always commute $\underset{\sim}{\Lambda}\underset{\sim}{\Omega} \neq \underset{\sim}{\Omega}\underset{\sim}{\Lambda}$ sometimes.
• real when: $\alpha = \alpha^*$	• Hermitian when $\underset{\sim}{\Omega} = \underset{\sim}{\Omega}^\dagger$
• imaginary when: $\alpha = -\alpha^*$	• Anti-Hermitian when $\underset{\sim}{\Omega} = -\underset{\sim}{\Omega}^\dagger$
• unit magnitude $\alpha \cdot \alpha^* = \alpha^*\alpha = 1$	• Unitary when $\underset{\sim}{\Omega}\underset{\sim}{\Omega}^\dagger = \underset{\sim}{\mathbb{1}} = \underset{\sim}{\Omega}^\dagger\underset{\sim}{\Omega}$
• Any complex number α: $\alpha = \underbrace{\tfrac{1}{2}(\alpha + \alpha^*)}_{\text{Real}} + \underbrace{\tfrac{1}{2}(\alpha - \alpha^*)}_{\text{Imaginary}}$	• Any operator $\underset{\sim}{\Omega}$: $\underset{\sim}{\Omega} = \underbrace{\tfrac{1}{2}(\underset{\sim}{\Omega} + \underset{\sim}{\Omega}^\dagger)}_{\text{Hermitian}} + \underbrace{\tfrac{1}{2}(\underset{\sim}{\Omega} - \underset{\sim}{\Omega}^\dagger)}_{\text{anti-Hermitian}}$

Unitary Transformations

$$\underset{\sim}{U}\underset{\sim}{U}^\dagger = \underbrace{\underset{\sim}{U}^\dagger\underset{\sim}{U}}_{\text{columns are unit vectors}} = \underset{\sim}{\mathbb{1}}$$

saying:

$$\begin{pmatrix} U_{11}^* & U_{21}^* & \cdots & U_{n1}^* \\ U_{12}^* & U_{22}^* & & \\ \vdots & & & \\ U_{1n}^* & & & \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} & U_{13} & \cdots & U_{1n} \\ U_{21} & U_{22} & U_{23} & & \\ U_{31} & U_{32} & U_{33} & & \\ \vdots & & & \ddots & \\ U_{n1} & & & & U_{nn} \end{pmatrix} = \begin{pmatrix} 1 & 0 & & & \\ 0 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

0's mean: they are orthogonal.